

POWER SERIES SOLUTIONS OF NON-LINEAR q -DIFFERENCE EQUATIONS AND THE NEWTON-PUISEUX POLYGON

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ABSTRACT. Adapting the Newton-Puiseux Polygon process to nonlinear q -difference equations of any order and degree, we compute their power series solutions, study the properties of the set of exponents of the solutions and give a bound for their q -Gevrey order in terms of the order of the original equation.

1. INTRODUCTION

The Newton Polygon construction for solving equations in terms of power series and its generalization by Puiseux has been successfully used countless times both in the algebraic [23], [24] [17] and in the differential contexts [15], [12, Ch. V], [18], [14], [7], [8], [10], [32] (this is just a biased and briefest of samples, see also [9] and [11, Sec. 29] for an interesting detailed historical narrative). We extend its use to q -difference equations.

Although this construction is primarily intended to give a method for computing formal power series solutions, we will use it for proving the q -analog of some results concerning the nature of power series solutions of non linear differential equations. Namely, we show properties about the growth of the coefficients of a power series solution (Maillet's theorem) and about the set of exponents of a generalized power series solution.

The method allows us, first of all, to show that the set of exponents of any generalized power series solution of a formal q -difference equation is finitely generated as a semigroup (in particular, it has finite rational rank and if the exponents are all rational, then their denominators are bounded). This mirrors the results of D. Y. Grigoriev and M. Singer in [14] for differential equations. When the q -difference equation is of first order and first degree, we give a bound for this rational rank (see Theorem 3 for a precise statement). We also study properties related to what we call "finite determination" (Definition 4) of the coefficients of the solutions. This is one of the places in which the case $|q| = 1$ is essentially different from the general case. For $|q| \neq 1$, we prove the finite determination of the coefficients.

Maillet's theorem [20] is a classical results about the growth of the coefficients a_i of a formal power series solution of a (non-linear) differential

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equation: it states that $|a_i| \leq i!^s R^i$, for some constants R and s . Among the different proofs (for instance [20, 21, 13]), B. Malgrange's [22] includes a precise bound for s . This bound is optimal except for one case: when the linearized operator along the solution has a regular singularity and the solution is a “non-regular solution”, for which any $s > 0$ works (see the last remark in Malgrange's paper); we shall refer to it as the (RS-N) case. In [7], the Newton Polygon method allows the author to prove Maillet's result and to show convergence (i.e. $s = 0$) in the (RS-N) case.

The first studies on convergence of solutions of non-linear q -difference equations are due to Bézivin [4], [3] and [5]. The q -analogue of Maillet's theorem states that when $|q| > 1$, a formal power series solution of a q -difference equation with analytic coefficients is q -Gevrey of some order s (see Definition 5). Zhang [33] proves this adapting Malgrange's proof to the case of q -difference-differential convergent equations. In this paper, the adaptation of the Newton Polygon to q -difference equations allow us to give a new proof of the q -analogue of Maillet's theorem and to extend it to the q -Gevrey non-convergent case. The bounds obtained for convergent equations match Zhang's in general and are more accurate in the (RS-N) case. However, we cannot prove convergence in this case unlike for differential equations.

Notice that the “Newton Polygon” construction used in the case of linear operators by Adams [1], Ramis [26], Sauloy [30] and others is different from the one presented here. In the linear case, the Newton Polygon is used to find local invariants of the operator while our Newton Polygon is constructed with the aim of looking for formal power series solutions. In Section 4 we describe the relation between Adams' Newton Polygon and Zhang's bounds. Adams' construction is also used in [19] to give conditions for the convergence of the solution(s) of analytic nonlinear q -difference equations.

For the reader's convenience, we include a final section with a detailed working example describing most of the constructions and the evolution of the Newton Polygon as one computes the successive terms of a solution.

2. THE NEWTON-PUISEUX POLYGON PROCESS FOR q -DIFFERENCE EQUATIONS

Let q be a nonzero complex number. For $j \in \mathbb{Z}$, let us denote by σ^j the automorphism of the ring $\mathbb{C}[[x]]$ of formal power series in one variable given by $\sigma^j(y(x)) = y(q^j x)$, that is,

$$\sigma\left(\sum_{i=0}^{\infty} a_i x^i\right) = \sum_{i=0}^{\infty} q^{ij} a_i x^i.$$

Let $P(x, Y_0, Y_1, \dots, Y_n) \in \mathbb{C}[[x, Y_0, \dots, Y_n]]$ be a formal power series. For $y \in \mathbb{C}[[x]]$, with $\text{ord}_x(y) > 0$, the expression $P(x, y, \sigma^1(y), \dots, \sigma^n(y))$ is a well-defined element of $\mathbb{C}[[x]]$ that we will be denoted by $P[y]$. We associate to $P(x, Y_0, Y_1, \dots, Y_n)$ the q -difference equation

$$(1) \quad P(x, y, \sigma^1(y), \dots, \sigma^n(y)) = 0.$$

We will look for solutions of equation (1) as formal power series with real exponents. We restrict ourselves to the Hahn field $\mathbb{C}((x^{\mathbb{R}}))$ of generalized power series, that is, formal power series of the form $\sum_{\gamma \in \mathbb{R}} c_{\gamma} x^{\gamma}$ whose support $\{\gamma \mid c_{\gamma} \neq 0\}$ is a well-ordered subset of \mathbb{R} and $c_{\gamma} \in \mathbb{C}$. Hahn fields were essentially introduced in [16]; see [29] for a detailed proof of the ring structure and [31] for a modern study in the context of functional equations. We fix a determination of the logarithm and extend the automorphism σ to $\mathbb{C}((x^{\mathbb{R}}))$ by setting

$$\sigma\left(\sum_{\gamma \in \mathbb{R}} c_{\gamma} x^{\gamma}\right) = \sum_{\gamma \in \mathbb{R}} q^{\gamma} c_{\gamma} x^{\gamma}.$$

For $y \in \mathbb{C}((x^{\mathbb{R}}))$, its order $\text{ord}(y)$ is the minimum of its support if $y \neq 0$ and $\text{ord}(0) = \infty$. In subsection 2.2, we shall see that if $\text{ord}(y) > 0$ then the expression $P(x, y, \sigma^1(y), \dots, \sigma^n(y))$ is a well-defined element of $\mathbb{C}((x^{\mathbb{R}}))$, hence equation (1) makes sense in our setting.

Although we look for solution in the Hahn field, their support has some finiteness properties, as in the case for differential equations. We say that $y \in \mathbb{C}((x^{\mathbb{R}}))$ is a grid-based series if there exists $\gamma_0 \in \mathbb{R}$ and a finitely generated semigroup $\Gamma \subseteq \mathbb{R}_{\geq 0}$ such that the support of y is contained in $\gamma_0 + \Gamma$. Puiseux series are the particular case of grid-based series in which $\gamma_0 \in \mathbb{Q}$ and $\Gamma \subseteq \mathbb{Q}$. Puiseux series and grid-based series form subfields of the Hahn field denoted respectively by $\mathbb{C}((x^{\mathbb{Q}}))^g$ and $\mathbb{C}((x^{\mathbb{R}}))^g$. We have

$$\mathbb{C}[[x]] \subseteq \mathbb{C}((x^{\mathbb{Q}}))^g \subseteq \mathbb{C}((x^{\mathbb{R}}))^g \subseteq \mathbb{C}((x^{\mathbb{R}})).$$

If equation (1) is algebraic, i.e. of the form $P(x, y) = 0$, then by Puiseux's Theorem all its formal power series solutions are of Puiseux type. This is no longer true if instead of \mathbb{C} , the base field is of positive characteristic, as the following example (due essentially to Ostrowski) shows: the equation $-y^p + xy + x = 0$ over the field $\mathbb{Z}/p\mathbb{Z}$ has as solution the generalized power series $y = \sum_{i=1}^{\infty} x^{\mu_i}$ with $\mu_i = (p^i - 1)/(p^{i+1} - p^i)$. Notice that the exponents are rational but they do not have a common denominator and moreover $\mu_1 < \mu_2 < \dots < 1/(p-1)$ so that they do not even go to infinity. Hence y is neither a Puiseux series nor a grid-based series.

As in the case of differential equations, the number of generalized power series solutions of a given equation (1) is not necessary finite, neither all of its solutions are of Puiseux type. For instance, the q -difference equation $Y_0 Y_2 - Y_1^2 = 0$ has $c x^{\mu}$ as solutions for any $c \in \mathbb{C}$ and $\mu \in \mathbb{R}$.

2.1. The Newton Polygon. Let $\mathcal{R} = \mathbb{C}[[x^{\mathbb{R}}]]$ be the subring of generalized power series with non-negative order. For a finitely generated semigroup of $\Gamma \subset \mathbb{R}_{\geq 0}$, the ring $\mathbb{C}[[x^{\Gamma}]]$ formed by those generalized power series with support contained in Γ is denoted by \mathcal{R}_{Γ} . Let $P \in \mathcal{R}[[Y_0, Y_1, \dots, Y_n]]$ be a nonzero formal power series in $n+1$ variables over \mathcal{R} . For $\rho = (\rho_0, \rho_1, \dots, \rho_n) \in \mathbb{N}^{n+1}$, we shall write $Y^{\rho} = Y_0^{\rho_0} \cdot Y_1^{\rho_1} \dots Y_n^{\rho_n}$; we shall also write $\mathcal{R}[[Y]]$ instead of $\mathcal{R}[[Y_0, Y_1, \dots, Y_n]]$. The coefficient of Y^{ρ} in P

will be denoted $P_\rho(x) \in \mathcal{R}$ and, for $\alpha \in \mathbb{R}$, the coefficient of x^α in $P_\rho(x)$ will be denoted $P_{\alpha,\rho} \in \mathbb{C}$, so that

$$P = \sum_{\rho \in \mathbb{N}^{n+1}} P_\rho(x) Y^\rho, \quad \text{and} \quad P_\rho(x) = \sum_{\alpha \in \Gamma_\rho} P_{\alpha,\rho} x^\alpha,$$

where for each ρ , Γ_ρ is a well-ordered subset of $\mathbb{R}_{\geq 0}$. We associate to P its *cloud of points* $\mathcal{C}(P)$: the set of points $(\alpha, j) \in \mathbb{R}^2$ for which there exists $\rho = (\rho_0, \dots, \rho_n)$ with $j = |\rho| = \rho_0 + \rho_1 + \dots + \rho_n$ and $P_{\alpha,\rho} \neq 0$.

The *Newton Polygon* $\mathcal{N}(P)$ of P is the convex hull of

$$\bar{\mathcal{C}}(P) = \{(\alpha + r, j) \mid (\alpha, j) \in \mathcal{C}(P), r \in \mathbb{R}_{\geq 0}\}.$$

A *supporting line* L of $\bar{\mathcal{C}}(P)$ is a line such that $\bar{\mathcal{C}}(P)$ is contained in the closed right half-plane defined by L and $L \cap \bar{\mathcal{C}}(P)$ is not empty, that is a line meeting $\mathcal{N}(P)$ on its border.

It will be convenient to speak about the *co-slope* of a line as the opposite of the inverse of its slope, the co-slope of a vertical line being 0. In order to deal with the particular case in which P is a polynomial in the variables Y_0, Y_1, \dots, Y_n we shall make use of the symbol $\mu_{-1}(P)$, denoting $-\infty$ if P is a polynomial and 0 otherwise. Hence the statement “ $\mu > \mu_{-1}(P)$ ” means “either $\mu > 0$ or P is a polynomial”.

Lemma 1. *Let $P \in \mathcal{R}[[Y]]$. For any $\mu > \mu_{-1}(P)$ there exists a unique supporting line of $\bar{\mathcal{C}}(P)$ with co-slope μ and the Newton polygon $\mathcal{N}(P)$ has a finite number of sides with co-slope greater or equal than μ . If P is a polynomial then $\mathcal{N}(P)$ has a finite number of sides and vertices. If $P \in \mathcal{R}_\Gamma[[Y]]$ for some finitely generated semigroup $\Gamma \subseteq \mathbb{R}_{\geq 0}$, then the Newton Polygon $\mathcal{N}(P)$ has a finite number of sides with positive co-slope.*

The unique supporting line with co-slope μ will be denoted henceforward $L(P; \mu)$.

Proof. If P is a polynomial, let h be its total degree the variables Y_0, \dots, Y_n . Otherwise we define h as follows: since $P \neq 0$ the set $\mathcal{C}(P)$ is nonempty; take a point $q \in \mathcal{C}(P)$ and let L be the line passing through q with co-slope μ . Let $(0, h)$ be the intersection of L with the OY -axis. For each $\rho \in \mathbb{N}^{n+1}$, write $\alpha_\rho = \text{ord } P_\rho(x)$. Only the finite number of points $(\alpha_\rho, |\rho|)$ with $|\rho| \leq h$ are relevant for the definition of the line $L(P; \mu)$ and for the construction of sides with co-slope greater or equal than μ of $\mathcal{N}(P)$. This proves the two first statements, the last one is a consequence of the fact that for a given $\alpha > 0$, the set $\Gamma \cap \{r < \alpha\}$ is finite. \square

For $\mu > \mu_{-1}(P)$, define the following polynomial in the variable C :

$$\Phi_{(P;\mu)}(C) = \sum_{(\alpha, |\rho|) \in L(P;\mu)} P_{\alpha,\rho} q^{\mu w(\rho)} C^{|\rho|},$$

where $w(\rho) = \rho_1 + 2\rho_2 + \dots + n\rho_n$. For a vertex v of $\mathcal{N}(P)$, the indicial polynomial is

$$\Psi_{(P;v)}(T) = \sum_{(\alpha, |\rho|)=v} P_{\alpha, \rho} T^{w(\rho)}.$$

2.2. Composition. For $s_0, \dots, s_n \in \mathbb{C}((x^{\mathbb{R}}))$, the expression $P(s_0, \dots, s_n)$ can be given a precise meaning under certain conditions. We consider on $\mathbb{C}((x^{\mathbb{R}}))$ the topology induced by the distance $d(f, g) = \exp(-\text{ord}(f - g))$ which is a complete topology.

If P is a polynomial, $P(s_0, \dots, s_n)$ is well-defined because $\mathbb{C}((x^{\mathbb{R}}))$ is a ring. Otherwise, we impose $\text{ord}(s_i) > 0$, for all i . Let $\mu = \min_{0 \leq i \leq n} \{\text{ord}(s_i)\}$. For $M \in \mathbb{N}$, consider the polynomial $P_{\leq M} = \sum_{|\rho| \leq M} P_{\rho}(x) Y^{\rho}$. The sequence $P_{\leq M}(s_0, \dots, s_n)$, $M \in \mathbb{N}$, is a Cauchy sequence because the order of $P_{\rho}(x) s_0^{\rho_0} \dots s_n^{\rho_n}$ is greater than or equal to $\mu |\rho|$. Its limit is precisely $P(s_0, \dots, s_n)$. Notice that if $P \in \mathcal{R}_{\Gamma}[[Y]]$ and all $s_i \in \mathcal{R}_{\Gamma}$, then $P(s_0, \dots, s_n) \in \mathcal{R}_{\Gamma}$.

Given s_0, \dots, s_n as above, we define the series

$$(2) \quad P(s_0 + Y_0, \dots, s_n + Y_n) := \sum_{\rho \in \mathbb{N}^{n+1}} \frac{1}{\rho!} \frac{\partial^{|\rho|} P}{\partial Y^{\rho}}(s_0, \dots, s_n) Y^{\rho},$$

where $\rho! = \rho_0! \dots \rho_n!$ and $\frac{\partial^{|\rho|} P}{\partial Y^{\rho}} = \frac{\partial^{|\rho|} P}{\partial Y_0^{\rho_0} \partial Y_1^{\rho_1} \dots \partial Y_n^{\rho_n}}$. For generalized power series $\bar{s}_0, \dots, \bar{s}_n$ with positive order it is straightforward to prove that the evaluation of the right hand side of (2) at $\bar{s}_0, \dots, \bar{s}_n$ is $P(s_0 + \bar{s}_0, \dots, s_n + \bar{s}_n)$.

If $y \in \mathbb{C}((x^{\mathbb{R}}))$ has $\text{ord}(y) > \mu_{-1}(P)$, then $P(y, \sigma(y), \dots, \sigma^n(y))$ is well defined because $\text{ord}(\sigma^k(y)) = \text{ord}(y)$. We also remark that if $y \in \mathcal{R}_{\Gamma}$, then $\sigma^k(y) \in \mathcal{R}_{\Gamma}$. The following notations will be used in the rest of the paper:

$$(3) \quad \begin{aligned} P[y] &= P(y, \sigma(y), \dots, \sigma^n(y)), \\ P[y + Y] &= P(y + Y_0, \sigma(y) + Y_1, \dots, \sigma^n(y) + Y_n). \end{aligned}$$

We are also going to make use of the little-o notation: $o(x^{\mu})$ will mean a generalized formal power series with order greater than μ or the zero series if $\mu = \infty$. The following is essentially what motivates the Newton polygon construction:

Lemma 2. *Let $y = cx^{\mu} + o(x^{\mu}) \in \mathbb{C}((x^{\mathbb{R}}))$, and $\mu > \mu_{-1}(P)$. Let $(\nu, 0)$ be the intersection point of $L(P; \mu)$ with the OX -axis. Then*

$$P[y] = \Phi_{(P; \mu)}(c) x^{\nu} + o(x^{\nu}),$$

In particular, if y is a solution of the q -difference equation (1) then

$$\Phi_{(P; \mu)}(c) = 0.$$

Proof. We may assume that $P = P_{\leq M}$ for some M , otherwise let $M \in \mathbb{N}$ such that $M\mu > \nu$. The truncation of $P[y]$ up to order ν is equal to that of $P_{\leq M}[y]$ and also $\Phi_{(P; \mu)}(C) = \Phi_{(P_{\leq M}; \mu)}(C)$.

Write $\alpha_\rho = \text{ord } P_\rho$, and notice that ν is the minimum of $\alpha_\rho + \mu|\rho|$, for $\rho \in \mathbb{N}^{n+1}$. The following chain of equalities proves the result

$$\begin{aligned}
 P_{\leq M}[cx^\mu + o(x^\mu)] &= \\
 \sum_{|\rho| \leq M} P_\rho(x) (cx^\mu + o(x^\mu))^{\rho_0} (q^\mu cx^\mu + o(x^\mu))^{\rho_1} \cdots (q^{n\mu} cx^\mu + o(x^\mu))^{\rho_n} &= \\
 \sum_{|\rho| \leq M} \{P_{\alpha_\rho, \rho} x^{\alpha_\rho} + o(x^{\alpha_\rho})\} \{c^{|\rho|} q^{\mu w(\rho)} x^{\mu|\rho|} + o(x^{\mu|\rho|})\} &= \\
 \sum_{|\rho| \leq M} \{P_{\alpha_\rho, \rho} c^{|\rho|} q^{\mu w(\rho)} x^{\alpha_\rho + \mu|\rho|} + o(x^{\alpha_\rho + \mu|\rho|})\} &= \\
 \left\{ \sum_{\alpha_\rho + \mu|\rho| = \nu} P_{\alpha_\rho, \rho} c^{|\rho|} q^{w(\rho)} \right\} x^\nu + o(x^\nu) &= \Phi_{(P; \mu)}(c) + o(x^\nu).
 \end{aligned}$$

The last equality holds because $L(P; \mu) = \{(\alpha, b) \mid \alpha + \mu b = \nu\}$. \square

Let $y \in \mathbb{C}((x^\mathbb{R}))$ be a generalized power series and S be its support. If S is finite, denote by $\omega(y)$ the cardinal of S , otherwise $\omega(y) = \infty$. Consider the sequence $\mu_i \in S$ defined inductively as follows: μ_0 is the minimum of S and for $0 \leq i < \omega(y)$, μ_{i+1} is the minimum of $S \setminus \{\mu_0, \mu_1, \dots, \mu_i\}$. Let $c_i \in \mathbb{C}$ be the coefficient of x^{μ_i} in y .

Definition 1. We shall call the first ω terms of y to the generalized power series $\sum_{0 \leq i < \omega(y)} c_i x^{\mu_i}$.

Notice that if the support of y is finite or has no accumulation points then y coincides with its first ω terms.

Corollary 1. Let y be a solution of the q -difference equation (1) and let $\sum_i c_i x^{\mu_i}$ be the first ω terms of y . Let P_i be the series defined as:

$$P_0 := P, \quad \text{and} \quad P_{i+1} := P_i[c_i x^{\mu_i} + Y], \quad 0 \leq i < \omega(y).$$

Then, for all $0 \leq i < \omega(y)$, one has

$$\Phi_{(P_i; \mu_i)}(c_i) = 0, \quad \text{and} \quad \mu_{i-1} < \mu_i,$$

where we denote $\mu_{-1} = \mu_{-1}(P)$.

Proof. Let $\bar{y}_k = y - \sum_{i=0}^{k-1} c_i x^{\mu_i}$, then $P_k[\bar{y}_k] = 0$ and the first term of \bar{y}_k is $c_k x^{\mu_k}$. \square

Let $P \in \mathcal{R}_\Gamma[[Y]]$ and let $\sum_{i=0}^\infty c_i x^{\mu_i}$ be a series with $\mu_{-1}(P) < \mu_i < \mu_{i+1}$, for all $0 \leq i < \infty$ (We do not impose that $c_i \neq 0$, but the sequence $(\mu_i)_{i \in \mathbb{N}}$ is strictly increasing). Consider the series $P_0 := P$ and $P_{i+1} := P_i[c_i x^{\mu_i} + Y]$.

Definition 2. We say that $\sum_{i=0}^\infty c_i x^{\mu_i}$ satisfies the necessary initial conditions for P , in short $\text{NIC}(P)$, if $\Phi_{(P_i; \mu_i)}(c_i) = 0$, for all $i \geq 0$.

The above Corollary states that the first ω terms of a solution of $P[y] = 0$ satisfy $\text{NIC}(P)$. In this section and the next one we shall prove in Proposition 2 the reciprocal statement for $P \in \mathcal{R}_\Gamma[[Y]]$: if $\sum_{i=0}^\infty c_i x^{\mu_i}$ satisfies $\text{NIC}(P)$, then $\lim_{i \rightarrow \infty} \mu_i = \infty$ and $\sum_{i=0}^\infty c_i x^{\mu_i}$ is an actual solution of the q -difference equation $P[y] = 0$. This implies in particular that solutions of $P[y] = 0$ coincide with their first ω terms.

A method for computing all the series satisfying $\text{NIC}(P)$ with $c_i \neq 0$, for all i , is the following one:

Procedure 1 (Computation of a power series satisfying $\text{NIC}(P)$).

Set $P_0 := P$ and $\mu_{-1} := \mu_{-1}(P)$.

For $i = 0, 1, 2, \dots$ do either (a.1) or (a.2) and (b), where:

(a.1). If $y = 0$ is a solution of $P_i[y] = 0$, then **return** $\sum_{k=0}^{i-1} c_k x^{\mu_k}$.

(a.2). Choose $\mu_i > \mu_{i-1}$, and $0 \neq c_i \in \mathbb{C}$ satisfying $\Phi_{(P_i, \mu_i)}(c_i) = 0$.

If neither (a.1) nor (a.2) can be performed then **return fail**.

(b). Set $P_{i+1}(Y) := P_i[c_i x^{\mu_i} + Y]$.

If **fail** is returned at step k of the above Procedure, this means that there are no solutions of $P[y] = 0$ having $\sum_{i=0}^{k-1} c_i x^{\mu_i}$ as its first k terms. To prove this, assume that z is a solution having $\sum_{i=0}^{k-1} c_i x^{\mu_i}$ as its first k terms. Either $z = \sum_{i=0}^{k-1} c_i x^{\mu_i}$, in which case $y = 0$ would be a solution of $P_k[y] = 0$ and (a.1) would have been performed, or $z - \sum_{i=0}^{k-1} c_i x^{\mu_i}$ would have a first term of the form $c_k x^{\mu_k}$ so that (a.2) could have been performed.

In order to carry out (a.2) in the above Procedure, one has to deal with the following formula with quantifiers

$$(4) \quad \exists \mu > \mu', \exists c \in \mathbb{C}, c \neq 0, \quad \Phi_{(P; \mu)}(c) = 0.$$

The Newton Polygon provides a way to eliminate the quantifiers. Fix $\mu' > \mu_{-1}(P)$; by Lemma 1, $\mathcal{N}(P)$ has only a finite number of sides L_1, L_2, \dots, L_t with co-slopes greater than μ' . Let $\gamma_1 < \gamma_2 < \dots < \gamma_t$ be their respective co-slopes and denote by v_{i-1} and v_i the endpoints of L_i . Take $\mu > \mu'$. Either $\mu = \gamma_j$ for some $1 \leq j \leq t$, or $\gamma_j < \mu < \gamma_{j+1}$ for some $0 \leq j \leq t$ (writing $\gamma_0 = \mu'$ and $\gamma_{t+1} = \infty$). If $\mu = \gamma_j$, then $L(P; \mu) \cap \mathcal{N}(P) = L_j$ and $\Phi_{(P; \mu)}(C)$ depends only on the coefficients $P_{\alpha, \rho}$ of P with $(\alpha, |\rho|) \in L_j$. Otherwise, $\gamma_j < \mu < \gamma_{j+1}$ for some j and $L(P; \mu) \cap \mathcal{N}(P)$ is just the vertex $v_j = (a, b)$, so that $\Phi_{(P; \mu)}(C) = C^b \cdot \Psi_{(P; v_j)}(q^\mu)$. Thus, there exists $c \neq 0$ and μ with $\gamma_j < \mu < \gamma_{j+1}$ such that $\Phi_{(P; \mu)}(c) = 0$ if and only if there exists μ , with $\gamma_j < \mu < \gamma_{j+1}$ and $\Psi_{(P; v_j)}(q^\mu) = 0$. This proves that equation (4) is equivalent to the quantifier-free formula obtained by the disjunction of the following formulæ:

$$(5) \quad \Phi_{(P; \gamma_j)}(c) = 0, \quad 1 \leq j \leq t,$$

$$(6) \quad \Psi_{(P; v_j)}(T) = 0, \mu = \log T / \log q, \gamma_j < \mu < \gamma_{j+1}, \quad 0 \leq j \leq t.$$

2.3. The pivot point. For $P \in \mathcal{R}_\Gamma[[Y]]$ and $\mu > \mu_{-1}(P)$, we shall denote by $Q(P; \mu)$ the point with highest ordinate in $L(P; \mu) \cap \mathcal{N}(P)$. For $\bar{P} =$

$P[cx^\mu + Y]$ (as in equation (3)), the following Lemma describes the Newton Polygon of \bar{P} :

Lemma 3. *Let h be the ordinate of $Q(P; \mu)$ and consider the half-planes $h^+ = \{(a, b) \in \mathbb{R}^2 \mid b \geq h\}$, $h^- = \{(a, b) \in \mathbb{R}^2 \mid b \leq h\}$. If $L(P; \mu)^+$ is the closed right half plane defined by $L(P; \mu)$ and $(\nu, 0)$ is the intersection of $L(P; \mu)$ with the OX -axis, then*

- (1) $\mathcal{N}(\bar{P}) \cap h^+ = \mathcal{N}(P) \cap h^+$, in particular $Q(P; \mu) \in \mathcal{N}(\bar{P})$. Moreover, for any α and ρ with $(\alpha, |\rho|) = Q(P; \mu)$, the coefficients $P_{\alpha, \rho}$ and $\bar{P}_{\alpha, \rho}$ are equal.
- (2) $\mathcal{N}(\bar{P}) \cap h^- \subseteq L(P; \mu)^+ \cap h^-$,
- (3) The point $(\nu, 0) \in \mathcal{N}(\bar{P})$ if and only if $\Phi_{(P; \mu)}(c) \neq 0$.

Proof. Write $M_\rho(Y) = P_\rho(x)Y^\rho$ and $\alpha_\rho = \text{ord } P_\rho(x)$. It is straightforward to show that $M_\rho[cx^\mu + Y] = M_\rho(Y) + V(Y)$ for some $V(Y)$, whose cloud of points is contained in the set $A_\rho = \{(a, b) \mid b < |\rho|\} \cap L(M_\rho; \mu)^+$. This proves part (2). If $Q = (\alpha, \rho)$ belongs to $\mathcal{N}(P) \cap h^+$, then there are no points $Q' = (\alpha', \rho') \in \mathcal{N}(P)$, except Q itself, such that $Q \in A_{\rho'}$. This proves part (1). Part (3) is a consequence of Lemma 2. \square

Corollary 2. *Let $\bar{\mu} > \mu$. Then either $Q(P; \mu) = Q(\bar{P}, \bar{\mu})$ or the ordinate of $Q(\bar{P}, \bar{\mu})$ is less than the ordinate of $Q(P; \mu)$. If $\Phi_{(P; \mu)}(c) \neq 0$, then the ordinate of $Q(\bar{P}, \bar{\mu})$ is zero.*

Proof. The previous Lemma implies that $Q(P; \mu)$ is a vertex of $\mathcal{N}(\bar{P})$ and $L(P; \mu) = L(\bar{P}; \mu)$. Hence $Q(P; \mu) = Q(\bar{P}; \mu)$. Since $\bar{\mu} > \mu$, $Q(\bar{P}; \bar{\mu})$ is a vertex with ordinate less than or equal to the ordinate of $Q(\bar{P}; \mu) = Q(P; \mu)$. For the second part, assume that $\Phi_{(P; \mu)}(c) \neq 0$. By the same Lemma, the point $(\nu, 0) \in \mathcal{N}(\bar{P})$, so that the segment whose endpoints are $(\nu, 0)$ and $Q(\bar{P}; \mu)$ is the only side of $\mathcal{N}(\bar{P})$ with co-slope greater than or equal to μ , from which follows that $Q(\bar{P}; \bar{\mu}) = (\nu, 0)$. \square

Let $P \in \mathcal{R}_\Gamma[[Y]]$ and take a series $\psi(x) = \sum_{i=0}^{\infty} c_i x^{\mu_i}$ with $\mu_{-1}(P) < \mu_i < \mu_{i+1}$ for all $0 \leq i < \infty$. (Notice that we do not impose that $c_i \neq 0$, but the sequence $(\mu_i)_{i \in \mathbb{N}}$ must be strictly increasing). Writing $P_0 := P$ and $P_{i+1} := P_i[c_i x^{\mu_i} + Y]$, let $Q_i = Q(P_i; \mu_i)$. By the previous Corollary, the ordinate of Q_i is less than or equal to the ordinate of Q_{i+1} . Since these are natural numbers, there exists N such that for $i \geq N$, the ordinate of Q_i is equal to the ordinate of Q_N (it stabilizes). By the same Corollary, we know that actually $Q_N = Q_i$, for all $i \geq N$. This leads to the following

Definition 3. *The pivot point of P with respect to $\psi(x)$ is the point Q at which the sequence Q_i stabilizes and is denoted by $Q(P; \psi(x))$. We say that it is reached at step N if $Q_N = Q(P; \psi(x))$.*

Let $Q_N = (\alpha, h)$ be the pivot point just defined. From part (1) of Lemma 3 follows that $(P_N)_{\alpha, \rho} = (P_i)_{\alpha, \rho}$ for all $i \geq N$, and for all ρ with $|\rho| = h$. In particular, the indicial polynomials $\Psi_{(P_i; Q_N)}(T)$ are the same for all $i \geq N$.

We shall say that *the monomial Y^ρ (resp. the variable Y_j) appears effectively in the pivot point* if $(P_N)_{\alpha,\rho} \neq 0$ (resp. for some ρ with $\rho_j > 0$).

Lemma 4. *Let P and $\psi(x) = \sum_{i=0}^{\infty} c_i x^{\mu_i}$ be as above. The following statements are equivalent:*

- (1) *The ordinate of the pivot point of P with respect to $\sum_{i=0}^{\infty} c_i x^{\mu_i}$ is greater than or equal to one.*
- (2) *The series $\sum_{i=0}^{\infty} c_i x^{\mu_i}$ satisfies $\text{NIC}(P)$.*

In case $\lim \mu_i = \infty$, these statements are equivalent to

- (3) *The series $\psi(x)$ is a solution of $P[y] = 0$.*

Proof. Assume statement (1). The ordinate of Q_{i+1} is non-zero and by the above Corollary, $\Phi_{(P_i;\mu_i)}(c_i) = 0$, which proves (2). Assume now that statement (1) is false, so that the ordinate of the pivot point is zero. This means that there exists some N such that Q_N has ordinate zero. By definition of Q_N we have that $L(P_N; \mu_N) \cap \mathcal{N}(P_N)$ is just the point $Q_N = (\alpha, 0)$. Then $\Phi_{(P_N;\mu_N)}(C)$ is a non-zero constant (namely the coefficient of x^α in P_N), therefore it has no roots, in contradiction with $\Phi_{(P_N;\mu_N)}(c_N) = 0$. This proves the equivalence between (1) and (2). By Corollary 1, (3) implies (2).

Assume (1) holds and that $\lim \mu_i = \infty$. Write $\psi_k(x) = \sum_{i=0}^{k-1} c_i x^{\mu_i}$ and notice that $P_i = P[\psi_i(x) + Y]$, in particular, $P[\psi_i(x)] = P_i[0] = (P_i)_0$. Let $Q = (\alpha, h)$ be the pivot point of P with respect to $\psi(x)$. Since $L(P_i; \mu_i)$ contains the point Q , $\text{ord}(P_i)_0 > \alpha + h\mu_i$ and since $h \geq 1$, the sequence $\text{ord } P[\psi_i(x)]$ tends to infinity and we are done. \square

Corollary 3. *Let $\sum_{i=0}^{\infty} c_i x^{\mu_i}$ be the first ω -terms of a solution of $P[y] = 0$. Then the pivot point of P with respect to $\sum_{i=0}^{\infty} c_i x^{\mu_i}$ has ordinate greater than or equal to one.*

2.4. Relative pivot points. The above construction of the pivot point can be made relative to any of the variables Y_j , $0 \leq j \leq n$, and more generally, relative to any monomial Y^r , with $r = (r_0, r_1, \dots, r_n) \in \mathbb{N}^{n+1}$, as follows:

Fix $r \in \mathbb{N}^{n+1}$. The cloud of points of P relative to Y^r is defined as the set $\mathcal{C}_r(P) = \{(\alpha, |\rho|) \mid \exists \rho, \text{ with } P_{\alpha,\rho} \neq 0, \text{ and } r \preceq \rho\}$, where $r \preceq \rho$ means that $r_i \leq \rho_i$, for all $0 \leq i \leq n$. It is obvious that $\mathcal{C}_r(P) \subseteq \mathcal{C}(P)$.

Assume that $\mathcal{C}_r(P)$ is not the empty set, then we may define the line $L_r(P; \mu)$ as the leftmost line with co-slope μ having nonempty intersection with $\mathcal{C}_r(P)$. The point $Q_r(P; \mu)$ will be the one with greatest ordinate in $L_r(P; \mu) \cap \mathcal{C}_r(P)$.

If H denotes $H = \frac{\partial^{|r|} P}{\partial Y^r}$, the cloud $\mathcal{C}_r(P)$ is not the empty set if and only if H is not the zero series. In this case, consider the translation map $\tau(a, b) = (a, b - |r|)$. It is straightforward to prove that $\mathcal{C}_r(P) = \tau^{-1}(\mathcal{C}(H))$. Hence, $L_r(P; \mu) = \tau^{-1}(L(H; \mu))$, and $Q_r(P; \mu) = \tau^{-1}(Q(H; \mu))$.

Let $\psi(x) = \sum_{i=0}^{\infty} c_i x^{\mu_i}$, with $\mu_0 > \mu_{-1}(P)$, and assume the notation of Definition 3. Denote also $H_0 = H$ and $H_{i+1} = H_i[c_i x^{\mu_i} + Y]$. By the chain

rule,

$$(7) \quad \frac{\partial^h P_i}{\partial Y^r} = H_i, \quad i \geq 0.$$

The sequence of points $Q_r(P_i; \mu_i) = \tau^{-1}(Q(H_i; \mu_i))$ for $i \geq 0$ stabilizes at some point denoted $Q_r(P; \psi(x))$ and which we call *the pivot point of P with respect to $\psi(x)$ relative to Y^r* . Therefore

$$(8) \quad Q(H; \psi(x)) = \tau(Q_r(P; \psi(x)))$$

Remark 1. Since $H \neq 0$, then $H_i \neq 0$, for $i \geq 0$ so that $\mathcal{C}_r(P_i)$ is not empty, for $i \geq 0$. This proves that $Q_r(P; \mu_i)$ and $Q_r(P; \psi(x))$ are well-defined provided the monomial Y^r appears effectively in P .

From now on, we shall denote e_j the vector $(0, \dots, 0, 1, 0, \dots, 0)$ where the 1 appears at position $j + 1$, for $j = 0, \dots, n$. Thus, $e_j = (\delta_{ij})_{0 \leq i \leq n} \in \mathbb{N}^{n+1}$ where δ_{ij} is the Kronecker symbol.

Proposition 1. *Let $Q = (a, h)$ be the pivot point of P with respect to $\psi(x)$. Assume that the monomial $Y^{r'}$ appears effectively in Q . Let $r \in \mathbb{N}^{n+1}$, with $r \preceq r'$, and $H = \frac{\partial^{|r|} P}{\partial Y^r}$. Then the pivot point of H with respect to $\psi(x)$ is $(a, h - |r|)$. In particular, if $r = r' - e_i$, for some i such that $r'_i \geq 1$, then the ordinate of the pivot point $Q(H; \psi(x))$ is one. However, for $r = r'$, one has $Q(H; \psi(x)) = (a, 0)$ and therefore $\psi(x)$ is not a solution of $H[y] = 0$.*

Proof. Assume the pivot point Q is reached at step N , thus $Q \in \mathcal{C}_{r'}(P_i) \subseteq \mathcal{C}_r(P_i)$ for all $i \geq N$. From $\mathcal{C}_r(P_i) \subseteq \mathcal{C}(P_i)$ and the fact that $Q = Q(P_i; \mu_i)$ for all $i > N$, one infers $Q = Q_r(P_i; \mu_i) = Q_{r'}(P_i; \mu_i)$ for all $i > N$. This means that Q is the pivot point of P with respect to $\psi(x)$ relative to Y^r and also relative to $Y^{r'}$. As we have seen before, $\tau_r(Q) = (a, h - |r|)$ is the pivot point of H with respect to $\psi(x)$. The third statement is a consequence of Lemma 4. \square

Corollary 4. *Let $\psi(x) = \sum_{i=0}^{\infty} c_i x^{\mu_i}$ be a solution of $P[y] = 0$ with $\lim \mu_i = \infty$. If the pivot point $(P; \psi(x))$ has ordinate greater than one, then there exists a non trivial derivative $H = \frac{\partial^{|r|} P}{\partial Y^r}$ of P , such that $\psi(x)$ is a solution of $H[y] = 0$.*

Proof. Let $Y^{r'}$ be a monomial that appears effectively in the pivot point $Q = Q(P; \psi(x))$. Since Q has ordinate greater than one, r' can be chosen with $|r'| \geq 2$. Let r be such that $r \preceq r'$ and $1 \leq |r| < |r'|$. By the Proposition, the pivot point of H with respect to $\psi(x)$ has ordinate greater than or equal to one. By Lemma 4, $\psi(x)$ is a solution of $H[y] = 0$. \square

Lemma 5. *Let $Q(P; \psi(x)) = (a, b)$ and $Q_r(P; \psi(x)) = (a', b')$ be respectively the general pivot point of $\psi(x)$ and the pivot point of $\psi(x)$ relative to Y^r . If the sequence μ_i of exponents of $\psi(x)$ tends to infinity, then $b' \geq b$ and if $b' = b$, then $a' \geq a$.*

Proof. Assume that both pivot points have been reached at step N . For any $i \geq N$, the point (a', b') belongs to the closed right half plane $L(P; \mu_i)^+$ because $\mathcal{C}_r(P_i) \subseteq \mathcal{C}(P_i)$. Since $(a, b) \in L(P; \mu_i)$ for all $i \geq N$, and $\lim \mu_i = \infty$, the intersection of all the half planes $L(P; \mu_i)^+$ for $i \geq N$, is the region formed by the points in $L(P; \mu_N)^+$ with ordinate greater than or equal to b . This proves the Lemma. \square

3. FINITENESS PROPERTIES

Throughout this section, we assume that Γ is a finitely generated semigroup of $\mathbb{R}_{\geq 0}$ and that P is a nonzero element of $\mathcal{R}_\Gamma[[Y]]$. We also assume that $q \neq 1$: the case $q = 1$ is reduced to the case $n = 0$ considering $P(Y_0, Y_0, \dots, Y_0)$. This section is devoted to proving the following results:

Theorem 1. *If $y \in \mathbb{C}((x^{\mathbb{R}}))$ is a solution of equation (1), then it is a grid-based formal power series.*

Proposition 2. *If $\psi(x) = \sum_{i=0}^{\infty} c_i x^{\mu_i}$ satisfies $\text{NIC}(P)$, then $\psi(x)$ is a solution of $P[y] = 0$.*

Definition 4. *Let $y \in \mathbb{C}((x^{\mathbb{R}}))$ and $P \in \mathcal{R}_\Gamma[[Y]]$. We say that y is finitely determined by P if there exist positive integers k and h , such that if y_k denotes the first k terms of y then y is the only element $z \in \mathbb{C}((x^{\mathbb{R}}))$ satisfying the following property: “ $z_k = y_k$ and $Q[y] = 0$ if and only if $Q[z] = 0$, for any $Q = \frac{\partial^{|r|} P}{\partial Y^r}$, with $|r| \leq h$.”*

Theorem 2. *If $|q| \neq 1$, then any solution y of equation (1) is finitely determined by P .*

The hypothesis $|q| \neq 1$ is necessary: let $P = Y_0 - Y_1$ and $q = \sqrt{-1}$. Any series $\sum_{i=0}^{\infty} c_{4i} x^{4i}$ (for arbitrary constants c_{4i}) is a solution of $P[y] = 0$. Since $\partial P / \partial Y_0[y] = 0$ and $\partial P / \partial Y_1[y] = 0$ have no solutions, and higher order derivatives of P are zero, none of these solutions is finitely determined by P .

To prove Theorem 1 we shall proceed as follows: let $\psi(x)$ be the first ω terms of y . By Corollary 1 above, $\psi(x)$ satisfies $\text{NIC}(P)$. We reduce the equation to what we call *quasi solved form* using Lemma 7 below and in this case, by Lemma 8, we infer that $\text{supp } \psi(x)$ is contained in a finitely generated semigroup of $\mathbb{R}_{\geq 0}$. By virtue of the following Remark 2, y coincides with $\psi(x)$. As a byproduct, we shall obtain a recursive formula for the coefficients of a solution which will be useful in the subsequent section.

Remark 2. Let Γ be a finitely generated semigroup of $\mathbb{R}_{\geq 0}$. For any real number k , the set $\Gamma \cap \{r \mid r \leq k\}$ is finite. Hence Γ is a well-ordered set with no accumulation points and its elements can be enumerated in increasing order: $\Gamma = \{\gamma_i\}_{i \geq 0}$, with $\gamma_i < \gamma_{i+1}$ and $\lim \gamma_i = \infty$. Let $\psi(x) = \sum_{i=0}^{\infty} c_i x^{\mu_i}$ be the first ω terms of an element $y \in \mathbb{C}[[x^{\mathbb{R}}]]$. If $\text{supp } \psi(x)$ is contained in Γ then either it is finite or $\lim \mu_i = \infty$. In both cases, $y = \psi(x)$. In particular, any element of $\mathbb{C}[[x^{\mathbb{R}}]]$ whose support is contained in Γ coincides with its first ω terms.

3.1. Quasi solved form. We say that the equation

$$(9) \quad P[y] = 0, \quad \text{ord}(y) > 0,$$

is in *quasi-solved form* if the point $(0, 1)$ is a vertex of $\mathcal{N}(P)$ and $(0, 0) \notin \mathcal{C}(P)$. If this is the case, let $\Psi(T)$ be the indicial polynomial of P at $(0, 1)$, $\Sigma = \{\mu \in \mathbb{R} \mid \Psi(q^\mu) = 0\}$ and $\Sigma^+ = \Sigma \cap \mathbb{R}_{>0}$. We say that equation (9) is in *solved form* if Σ^+ is the empty set.

Remark 3. The polynomial $\Psi(T)$ can be written $\Psi(T) = P_{0,e_0} + P_{0,e_1}T + \dots + P_{0,e_n}T^n \in \mathbb{C}[T]$. Its degree m is the largest index such that the variable Y_m appears effectively in the point $(0, 1)$. If the equation is in quasi-solved form, $\Psi(T)$ is a nonzero polynomial because $(0, 1) \in \mathcal{C}(P)$. If $|q| \neq 1$, then Σ is finite. If case $|q| = 1$ (and $q \neq 1$), then Σ is the finite union of the sets $\Sigma_r = \frac{\arg(r)}{\arg(q)} + \frac{2\pi}{\arg(q)}\mathbb{Z}$, for those complex roots r of $\psi(T)$ with modulus one. Recall that we have fixed a determination of the logarithm to compute q^μ , hence $\arg(q)$ is also fixed. The following Lemma implies that $\Sigma_r \cap \mathbb{R}_{\geq 0}$ is a finitely generated semigroup. Therefore Σ^+ generates a finitely generated semigroup of $\mathbb{R}_{\geq 0}$.

Lemma 6. *Let $\gamma \in \mathbb{R}$ and $\gamma_1, \gamma_2, \dots, \gamma_s$ positive real numbers. Then the semigroup Γ of $\mathbb{R}_{\geq 0}$ generated by the set $A = (\gamma + \gamma_1\mathbb{N} + \dots + \gamma_s\mathbb{N}) \cap \mathbb{R}_{\geq 0}$ is finitely generated.*

Proof. Let Λ be the set of $(n_1, \dots, n_s) \in \mathbb{N}^s$ such that $\gamma + \sum n_i \gamma_i > 0$. By Dickson's lemma, the number of minimal elements in Λ with respect the product order are finite. Hence Γ is generated by $\gamma_1, \dots, \gamma_s$ and the family $\gamma + \sum n_i \gamma_i > 0$ for all minimal element (n_1, \dots, n_s) of Λ . \square

3.1.1. Change of variable $z = x^\gamma y$. Let $P \in \mathcal{R}_\Gamma[[Y]]$ and $\gamma > \mu_{-1}(P)$. Define $P[x^\gamma Y]$ as the series

$$\sum_{\rho} q^{\gamma \omega(\rho)} x^{\gamma |\rho|} P_\rho(x) Y_0^{\rho_0} Y_1^{\rho_1} \dots Y_n^{\rho_n} \in \mathbb{C}((x^\mathbb{R}))^g [[Y]].$$

If $(\nu, 0)$ is the intersection point of $L(P; \gamma)$ with the OX -axis, then all the coefficients of the series $P[x^\gamma Y]$ have order greater than or equal to ν . Define $\gamma P = x^{-\nu} P[x^\gamma Y]$. The coefficients of γP are in \mathcal{R}_{Γ^*} , where Γ^* is the semigroup of $\mathbb{R}_{\geq 0}$ generated by $(-\nu + \Gamma + \gamma\mathbb{N}) \cap \mathbb{R}_{\geq 0}$. By Lemma 6, Γ^* is a finitely generated semigroup of $\mathbb{R}_{\geq 0}$.

The transformation $P \mapsto \gamma P$ corresponds to the change of variable $z = x^\gamma y$ in the following sense: for a series y , with $\text{ord } y > \gamma + \mu_{-1}(P)$, one has $\gamma P[x^{-\gamma} y] = x^{-\nu} P[y]$, in particular, $P[y] = 0$ if and only if $\gamma P[x^{-\gamma} y] = 0$.

Let $\bar{\tau}(a, b)$ be the plane affine map $\bar{\tau}(a, b) = (a - \nu + \gamma b, b)$, which satisfies $\bar{\tau}(\mathcal{C}_j(P)) = \mathcal{C}_j(\gamma P)$ for $0 \leq j \leq n$. In particular, $\bar{\tau}(\mathcal{N}(P)) = \mathcal{N}(\gamma P)$, and $\bar{\tau}$ maps vertices to vertices and sides of co-slope $\mu \geq \gamma$ to sides of co-slope $\mu - \gamma$. Moreover, $\bar{\tau}(L(P; \mu)) = L(\gamma P; \mu - \gamma)$, in particular $\bar{\tau}(L(P; \gamma)) = L(\gamma P; 0)$ is the vertical axis. Therefore, $Q(P; \mu)$ and $Q(\gamma P; \mu - \gamma)$ have the same ordinate. Let $\sum_{i=0}^{\infty} c_i x^{\mu_i}$ and P_i be as in the definition of pivot point

(Definition 3). Assume $\gamma < \mu_0$ and set $H = {}^\gamma P$, $H_0 = H$ and $H_{i+1} = H_i[c_i x^{\mu_i - \gamma} + Y]$. It is straightforward to prove that ${}^\gamma P_i = H_i$, so that $\bar{\tau}(Q(P_i; \mu_i)) = Q(H_i; \mu_i - \gamma)$ and, in particular, they have the same ordinate. Then the image by $\bar{\tau}$ of the pivot point of P with respect to $\sum_{i=0}^{\infty} c_i x^{\mu_i}$ is the pivot point of ${}^\gamma P$ with respect to $\sum_{i=0}^{\infty} c_i x^{\mu_i - \gamma}$ and the same holds for relative pivot points. By Lemma 4, this implies that $\sum_{i=0}^{\infty} c_i x^{\mu_i}$ satisfies $\text{NIC}(P)$ if and only if $\sum_{i=0}^{\infty} c_i x^{\mu_i - \gamma}$ satisfies $\text{NIC}({}^\gamma P)$.

Finally, if $v \in \mathcal{C}(P)$ then $\bar{\tau}(v) \in \mathcal{C}({}^\gamma P)$ and $\Psi_{({}^\gamma P; \bar{\tau}(v))}(T) = \Psi_{(P; v)}(q^\gamma T)$.

Lemma 7. *Assume that $\psi(x) = \sum_{i=0}^{\infty} c_i x^{\mu_i}$ satisfies $\text{NIC}(P)$. Then there exist a finitely generated semigroup Γ^* , a series $P^* \in \mathcal{R}_{\Gamma^*}[[Y_0, Y_1, \dots, Y_n]]$, an index N and a rational number γ with $\mu_{N-1} \leq \gamma < \mu_N$, such that the equation*

$$(10) \quad P^*[z] = 0, \quad \text{ord } z > 0$$

is in quasi solved form and $\psi^(x) = \sum_{i=N}^{\infty} c_i x^{\mu_i - \gamma}$ satisfies $\text{NIC}(P^*)$.*

Proof. We may assume that the ordinate of the pivot point of $\psi(x)$ with respect to P is one. Otherwise, by Proposition 1, we may replace P by any of its derivatives $\frac{\partial^{[r]} P}{\partial Y^r}$, where the monomial $Y_j Y^r$ appears effectively in the pivot point, for some j . We remark that the coefficients of any derivative of P also belong to \mathcal{R}_Γ . Let $Q = (\alpha, 1)$ be the pivot point of P with respect to $\psi(x)$ and use the notation of Definition 3: $P_0 = P$, $P_{i+1} = P_i[c_i x^{\mu_i} + Y]$ and so on. In particular, let the pivot point be reached at step $N' - 1$ for some N' . Consider any integer $N \geq N'$. Denote $\Gamma_0 = \Gamma$ and $\Gamma_{i+1} = \Gamma_i + \mu_i \mathbb{N}$. Notice that the coefficients of P_i belong to \mathcal{R}_{Γ_i} .

Let γ be a rational number such that $\mu_{N-1} \leq \gamma < \mu_N$ and set $P^* = {}^\gamma P_N \in \mathcal{R}_{\Gamma_N^*}[[Y]]$. Since the pivot point Q has been reached at step $N - 1$, $Q \in L(P_{N-1}; \mu_{N-1}) \cap L(P_N; \mu_N)$. By Proposition 3, $Q \in L(P_N; \mu_{N-1})$. Hence $Q \in L(P_N; \mu_{N-1}) \cap L(P_N; \mu_N)$; since $\mu_{N-1} < \gamma < \mu_N$, we conclude that $Q(P_N; \gamma) = Q = (\alpha, 1)$. So, by the properties described in 3.1.1, the point $(0, 1)$ is in $\mathcal{C}(P^*)$, the equation $P^*[y] = 0$ is in quasi solved form and the pivot point of P^* with respect to $\psi^*(x)$ is $(0, 1)$. By Lemma 4, $\psi^*(x)$ satisfies $\text{NIC}(P^*)$. \square

Lemma 8. *Assume equation (10) is in quasi-solved form and let $\xi(x) = \sum_{i=0}^{\infty} c_i x^{\mu_i}$, with $\mu_0 > 0$, be a series satisfying $\text{NIC}(P^*)$. Then the support of $\xi(x)$ is contained in the finitely generated semigroup $\Gamma' = \Gamma^* + \Sigma^+ \mathbb{N}$. In particular, either the support of $\xi(x)$ is finite or $\lim \mu_i = \infty$ and in both cases $\xi(x)$ is a solution of equation (10).*

Proof. Let $P_0 = P^*$ and $P_{i+1} = P_i[c_i x^{\mu_i} + Y]$ for $i \geq 0$. We first prove that $Q(P_i; \mu_i) = (0, 1)$ for all $i \geq 0$. We do this showing, by induction on i , that $\mathcal{N}(P_i)$ is contained into the first quadrant of the plane and that the point $(0, 1) \in \mathcal{C}(P_i)$. This holds for P_0 because of the hypotheses on P^* . Assume that the statement holds for P_i . Since $\mu_i > 0$, the line $L(P_i; \mu_i)$ either contains the point $(0, 1)$, and then $Q(P_i; \mu_i) = (0, 1)$, or $L(P_i; \mu_i)$

meets $\mathcal{N}(P_i)$ at a single point with zero ordinate which is $Q(P_i; \mu_i)$. If the latter happens, from Corollary 2, we infer that the pivot point of P^* with respect to $\xi(x)$ has zero ordinate, in contradiction with the fact that $\xi(x)$ satisfies $\text{NIC}(P^*)$. Hence $Q(P_i; \mu_i) = (0, 1)$. By Lemma 3, $(0, 1)$ is a vertex of $\mathcal{N}(P_{i+1})$ and since $P_{i+1} \in \mathcal{R}[[Y]]$, its Newton polygon is contained in the first quadrant. This proves the induction step and that $Q(P_i; \mu_i) = (0, 1)$, $i \geq 0$.

The fact that $Q(P_i; \mu_i) = (0, 1)$ implies that the polynomial $\Phi_{(P_i; \mu_i)}(C)$ is equal to $\Psi(q^{\mu_i})C + \text{Coeff}(P_i; x^{\mu_i} Y^0)$, where $\Psi(T)$ is the indicial polynomial of P at $(0, 1)$ and $\text{Coeff}(P_i; x^{\mu_i} Y^0)$ is the coefficient of x^{μ_i} of $Y_0^0 Y_1^0 \dots Y_n^0$ in P_i . Since $\Phi_{(P_i; \mu_i)}(c_i) = 0$ because $\xi(x)$ satisfies $\text{NIC}(P^*)$, the following equations hold:

$$(11) \quad \Psi(q^{\mu_i}) c_i + \text{Coeff}(P_i; x^{\mu_i} Y^0) = 0, \quad i \geq 0.$$

Let us prove, by induction, that $P_i \in \mathcal{R}_{\Gamma'}[[Y]]$, for all $i \geq 0$, and that the support of $\xi(x)$ is contained in Γ' . By hypothesis, $P_0 \in \mathcal{R}_{\Gamma'}[[Y]]$. Assume that $P_i \in \mathcal{R}_{\Gamma'}[[Y]]$. If $c_i = 0$, then $P_{i+1} = P_i \in \mathcal{R}_{\Gamma'}[[Y]]$ and $\mu_i \notin \text{supp}(\xi(x))$. If, on the contrary, $c_i \neq 0$, we can prove by contradiction that $\mu_i \in \Gamma'$: assume that $\mu_i \notin \Gamma'$, in particular $\mu_i \notin \Sigma^+$, hence $\Psi(q^{\mu_i}) \neq 0$. From equation (11), $\text{Coeff}(P_i; x^{\mu_i} Y^0) \neq 0$, we deduce that $\mu_i \in \text{supp}((P_i)_0) \subseteq \Gamma'$. So $P_{i+1} = P_i[c_i x^{\mu_i} + Y]$ belongs to $\mathcal{R}_{\Gamma'}[[Y]]$ which proves the induction step.

The set $\text{supp} \xi(x)$ has no accumulation points in \mathbb{R} because Γ' is a finitely generated semigroup of $\mathbb{R}_{\geq 0}$ and $\text{supp} \xi(x) \subseteq \Gamma'$ and we are done. \square

Corollary 5. *Let y be a solution of equation (10) which is in quasi solved form. Let $\Gamma' = \{\gamma_i\}_{i=0}^\infty$, with $\gamma_i < \gamma_{i+1}$ for all i . Then $y = \sum_{i=1}^\infty d_i x^{\gamma_i}$ with d_i satisfying the following recurrent formula:*

$$(12) \quad \Psi(q^{\gamma_i}) d_i = -\text{Coeff}(P^*[d_1 x^{\gamma_1} + \dots + d_{i-1} x^{\gamma_{i-1}}]; x^{\gamma_i}), \quad i \geq 1.$$

If Σ^+ is finite and z is another solution of equation (10) with $\text{ord}(y - z)$ greater than any element of Σ^+ , then $y = z$.

Proof. Let $\xi(x)$ be the first ω terms of y . Then $\text{supp} \xi(x) \subseteq \Gamma'$, and by Remark 2, $y = \xi(x) \in \mathcal{R}_{\Gamma'}$. Hence we may write $y = \sum_{i=1}^\infty d_i x^{\gamma_i}$ because $\gamma_0 = 0$ and $\text{ord} y > 0$. Since $\xi(x)$ satisfies $\text{NIC}(P^*)$, the same reasoning as in Lemma 8 up to equation (11) holds. The coefficient $\text{Coeff}(P_i; x^{\gamma_i} Y^0)$ is equal to the coefficient of x^{γ_i} in $P^*[d_1 x^{\gamma_1} + \dots + d_{i-1} x^{\gamma_{i-1}}]$, which gives equation (12). To prove the last statement, write $z = \sum_{i=1}^\infty d'_i x^{\gamma_i}$. If γ_i is greater than any element of Σ^+ , then $\Psi(q^{\gamma_i}) \neq 0$, and d_i is completely determined by d_1, \dots, d_{i-1} , so that $y = z$. \square

Proof of Proposition 2. Applying Lemma 7 to $\psi(x)$ we obtain equation (10), and applying Lemma 8 to $\xi(x) = \sum_{i=N}^\infty c_i x^{\mu_i - \gamma}$ we conclude that $\mu_i - \gamma \in \Gamma'$, for $i \geq N$. Since $\gamma \geq \mu_0$, the set $(\gamma - \mu_0) + \Gamma'$ is included in $\mathbb{R}_{\geq 0}$. Let Γ'' be the semigroup generated by $(\gamma - \mu_0) + \Gamma'$. By Lemma 6, Γ'' is finitely generated. Let Γ''' be the finitely generated semigroup $\Gamma'' + \sum_{i=0}^{N-1} (\mu_i - \mu_0) \mathbb{N}$. The set $\text{supp} \psi(x)$ is contained in $\mu_0 + \Gamma'''$, so that $\lim \mu_i = \infty$. By Lemma 4, $\psi(x)$ is a solution of $P[y] = 0$. \square

Proof of Theorem 1. Let $\psi(x) = \sum_{i=0}^{\infty} c_i x^{\mu_i}$ be the first ω terms of y . By Corollary 1, $\psi(x)$ satisfies $\text{NIC}(P)$. As in the proof of Proposition 2, there exists a finitely generated semigroup Γ such that $\text{supp } \psi(x)$ is contained in $\mu_0 + \Gamma$. By Remark 2, $y = \psi(x)$, so that y is grid-based. \square

Proof of Theorem 2. Let y be a solution of equation (1). By Theorem 1, y coincides with its first ω terms. Write $y = \sum_{i=0}^{\infty} c_i x^{\mu_i}$ and let $Q = (\alpha, h)$ be the pivot point of P with respect to y . Apply Lemmas 7 and 8 to y : let N and γ be as in Lemma 7; we may assume that the pivot point Q is reached at step $N - 1$. Since $|q| \neq 1$, Σ^+ is finite by Remark 3. Since $\lim \mu_i = \infty$, there is $k > N$ such that $\mu_k - \gamma$ is greater than any element of Σ^+ .

Consider $z \in \mathbb{C}((x^{\mathbb{R}}))$ with the same first k terms as y and satisfying that for any $H = \frac{\partial^{|r|} P}{\partial Y^r}$, with $|r| \leq h$, $H[y] = 0$ if and only if $H[z] = 0$. We have to show that $y = z$.

Since $P[y] = 0$, then $P[z] = 0$, and z coincides with its first ω terms. Write $z = \sum_{i=0}^{\infty} d_i x^{\delta_i}$. By hypothesis, $c_i = d_i$ and $\mu_i = \delta_i$ for $0 \leq i < k$. Denote $P'_0 = P$, $P'_{i+1} = P_i[d_i x^{\delta_i} + Y]$ and $P_0 = P$ and $P_{i+1} = P_i[c_i x^{\mu_i} + Y]$, for $i \geq 0$. Obviously, $P_i = P'_i$, for $0 \leq i \leq k$. In particular $Q = Q(P_N; \mu_N) = Q(P'_N; \delta_N)$.

If the pivot point of P with respect to z is also Q , then apply Lemmas 7 and 8 to z in the same way as to y : choose the same derivative $\frac{\partial^{h-1} P}{\partial Y^r}$, the same N and the same γ to obtain the same P^* . This can be done because $P_i = P'_i$, for $0 \leq i \leq k$. This implies that $\xi(x) = \sum_{i=N}^{\infty} c_i x^{\mu_i - \gamma}$ and $\bar{\xi}(x) = \sum_{i=N}^{\infty} d_i x^{\delta_i - \gamma}$ both satisfy $\text{NIC}(P^*)$. By Corollary 5, $\xi(x) = \bar{\xi}(x)$, which implies $y = z$.

Let us show by contradiction that the pivot point Q' of P with respect to z must be Q . Assume $Q' \neq Q$. Hence $Q = Q(P'_N; \delta_N)$ is not the stabilization point of the sequence $Q(P'_i; \delta_i)$. This implies that Q' has ordinate $h' < h$. Let Y^r be a monomial that appears effectively in the pivot point of P with respect to z , so that $|r| = h'$. Let $H = \frac{\partial^{h'} P}{\partial Y^r}$. By Proposition 1, $H[z] \neq 0$; in particular $H \neq 0$. We claim that $H[y] = 0$. By Remark 1, the pivot point Q_r of P with respect to y relative to Y^r is well-defined. Since $\lim \mu_i = \infty$, by Lemma 5, the ordinate of Q_r is $h'' \geq h$. The pivot point of H with respect to y has ordinate $h'' - h' \geq h - h' \geq 1$. By Lemma 4, y satisfies $\text{NIC}(P)$ and so $H[y] = 0$, which proves our claim and finishes the proof of the Theorem. \square

3.2. Bounding the rational rank in the case of order and degree one. Recall that the rational rank of a semigroup $S \subseteq \mathbb{R}$ is the dimension of $\langle S \rangle$, the \mathbb{Q} -vectorial subspace of \mathbb{R} generated by S . It is denoted $\text{rat. rk}(S)$.

In what follows Γ denotes a finitely generated semigroup of $\mathcal{R}_{\geq 0}$, as above.

Theorem 3. Assume $|q| \neq 1$. Let $P = A(Y_0) + B(Y_0)Y_1$ be a nonzero series, where $A, B \in \mathcal{R}_{\Gamma}[[Y_0]]$. Let y be a solution of $P[y] = 0$, with $\text{ord } y > \mu_{-1}(P)$. Then $\text{rat. rk}(\text{supp } y) \leq \text{rat. rk}(\Gamma) + 1$.

Proof. By the previous results, y coincides with its first ω terms $\psi(x) = \sum_{i=0}^{\infty} c_i x^{\mu_i}$. Taking a rational $\gamma < \mu_0$ and replacing P by γP we may assume that $\mu_0 > 0$ and that $\gamma P \in \mathcal{R}_{\Gamma^*}$ and $\text{rat.rk}(\Gamma^*) = \text{rat.rk}(\Gamma)$, for another finitely generated semigroup Γ^* .

Define $P_0 = P$, $P_{i+1} = P_i[c_i x^{\mu_i} + Y]$, $\Gamma_0 = \Gamma^*$ and $\Gamma_{i+1} = \Gamma_i + \mu_i \mathbb{N}$. The coefficients of P_i belong to \mathcal{R}_{Γ_i} . Notice that one has $\dim \langle \Gamma_{i+1} \rangle \leq \dim \langle \Gamma_i \rangle + 1$ and the inequality holds only if $\mu_i \notin \langle \Gamma_i \rangle$.

For each i , the line $L(P_i; \mu_i)$ corresponds either to a vertex or to a side of $\mathcal{N}(P_i)$. If it corresponds to a side, then there are two different points (α, a) and (β, b) in $\mathcal{C}(P_i)$ lying on $L(P_i; \mu_i)$. This implies that $\alpha, \beta \in \Gamma_i$ and $\mu_i = (\beta - \alpha)/(a - b) \in \langle \Gamma_i \rangle$. Hence it is enough to prove that if for an index i , μ_i corresponds to a vertex of $\mathcal{N}(P_i)$, then for all $j > i$, μ_j corresponds to a side of $\mathcal{N}(P_j)$. Assume then that μ_i corresponds to a vertex $v = (a, h)$ lying on $L(P_i; \mu_i)$.

We delay to the following Remark proving that $v' = Q(P_{i+1}; \mu_{i+1})$ has ordinate equal to one and that $\Psi_{(P_{i+1}; v')}(q^\mu) \neq 0$, for $\mu > \mu_i$.

We can thus write $v' = (a', 1)$, so that v' is the pivot point of P with respect to $\psi(x)$. We know that $a' \in \Gamma_{i+1}$ and that, for $j > i$,

$$0 = \Phi_{(P_j; \mu_j)}(c_j) = \Psi_{(P_j; v')}(q^{\mu_j})c_j + \text{Coeff}(P_j; x^{a' + \mu_j} Y^0).$$

Since $\Psi_{(P_j; v')}(q^{\mu_j}) = \Psi_{(P_{i+1}; v')}(q^{\mu_j}) \neq 0$ and $c_j \neq 0$, this implies $a' + \mu_j \in \Gamma_j$. Hence, $\mu_j \in \langle \Gamma_{i+1} \rangle$ for $j > i$, which proves that $\text{rat.rk}(\{\mu_l \mid l \geq 0\}) \leq \text{rat.rk}(\Gamma_{i+1}) \leq \text{rat.rk}(\Gamma_0) + 1$. \square

Remark 4. (Proof that the pivot point of P with respect to $\psi(x)$ has ordinate 1 and that $\Psi_{(P_{i+1}; v')}(q^\mu) \neq 0$, for $\mu \neq \mu_i$). We write μ , c and P instead of μ_i , c_i and P_i . Since $L(P; \mu) \cap \mathcal{N}(P) = \{v\}$, we have that $\Phi_{(P; \mu)}(c) = \Psi_{(P; v)}(q^\mu)c^h$. Let $M(Y_0, Y_1) = Ax^a Y_0^h + Bx^a Y_0^{h-1} Y_1$, $A, B \in \mathbb{C}$, be the sum of the terms of P corresponding to v . Then $\Psi_{(P; v)}(T) = A + BT$, so that $q^\mu = -A/B$. Let $\bar{P} = P[cx^\mu + Y]$ and $\bar{M} = M[cx^\mu + Y]$. By direct computation, $\bar{M} = c^{h-1} x^{a+\mu(h-1)} (AY_0 + BY_1) + M'$, where M' has only terms of total degree at least two in Y_0, Y_1 , so that the point $v' = (a + \mu(h-1), 1)$ is a vertex of $\mathcal{N}(\bar{P})$ and $\Psi_{(\bar{P}; v')}(T) = c^{h-1} \Psi_{(P; v)}(T)$. Since $|q| \neq 1$, for $\mu' > \mu$, $\Psi_{(\bar{P}; v')}(q^{\mu'}) \neq 0$.

4. q -GEVREY ORDER

Throughout this section we assume that $|q| > 1$. In this case, we prove some properties about the growth of the coefficients of a formal power series solution of a q -difference equation.

Definition 5. A formal power series $\sum_{i=0}^{\infty} c_i x^i$ is said to be of q -Gevrey order $s \geq 0$ if the series $\sum_{i=0}^{\infty} c_i |q|^{-\frac{1}{2}s i^2} x^i$ has a positive radius of convergence.

We will say that a series $P = \sum_{\alpha, \rho} P_{\alpha, \rho} x^\alpha Y^\rho \in \mathbb{C}[[x, Y_0, Y_1, \dots, Y_n]]$ is of q -Gevrey order $s \geq 0$ if the series

$$\sum_{(\alpha, \rho) \in \mathbb{N} \times \mathbb{N}^{n+1}} P_{\alpha, \rho} |q|^{-\frac{1}{2}s(\alpha+|\rho|)^2} x^\alpha Y^\rho$$

has a positive radius of convergence at the origin of \mathbb{C}^{n+2} .

We remark that q -Gevrey of order 0 means convergence. This section is devoted to proving the following result (the number $s(P; y(x))$ in the statement is introduced in Definition 6 and can be computed from the relative pivot points of P with respect to $y(x)$).

Theorem 4. *Let $P \in \mathbb{C}[[x, Y_0, Y_1, \dots, Y_n]]$ be a non-zero formal power series of q -Gevrey order $t \geq 0$ and $y(x) \in \mathbb{C}[[x]]$ a solution of $P[y] = 0$. Then $y(x)$ is of q -Gevrey order $t + s(P; y)$ (see the following definition).*

Definition 6. *Let $Q = (a, h)$ be the pivot point of P with respect to $y(x)$. The number $s(P; y)$ is defined as follows:*

Case $h = 1$. *Let $Q_j = (a_j, h_j)$ be the pivot point of P with respect to $y(x)$ relative to the variable Y_j (for $0 \leq j \leq n$). Since Q has ordinate 1, $Q = Q_j$ for some j . Let $r = \max\{j \mid Q_j = Q\}$. There are three cases:*

(RS-R) *If $r = n$, then $s(P; y(x)) = 0$.*

(RS-N) *If $r < n$ and $h_j > 1$ for all $r < j \leq n$, then $s(P; y(x))$ can be taken as any positive number.*

(IS) *If $r < n$ and $h_j = 1$ for some $r < j \leq n$, then $s(P; y(x)) = \max\{\frac{j-r}{a_j-a_r} \mid r < j \leq n, h_j = 1\}$.*

Case $h > 1$. *By Proposition 1 there exist derivatives $H = \frac{\partial |\rho| P}{\partial Y^\rho}$, with $|\rho| = h - 1$, such that the pivot point of H with respect to $y(x)$ is equal to 1. Define $s(P; y(x))$ as the minimum of all those $s(H; y(x))$.*

Remark 5. When Q has ordinate $h > 1$, the number $s(P; y(x))$ can be described directly in terms of the relative pivot points: let $Q = (a, h)$ be the (general) pivot point of P with respect to $y(x)$, and $Q_\rho(P; y(x)) = (a_\rho, h_\rho)$. Let A be the set formed by those 3-tuples (ρ, i, j) satisfying the following properties: $|\rho| = h$, $Q_\rho = Q$, $0 \leq i < j \leq n$, and $h_{\rho'} = h$, where $\rho' = \rho - e_i + e_j$. If the set A is empty, we define $s(P; y(x))$ as any positive real number. Otherwise, $s(P; y(x))$ is the minimum of $\frac{j-i}{a_{\rho'}-a}$, for those $(\rho, i, j) \in A$.

Remark 6. Zhang's paper [33] deals with the case in which P is a convergent series. The bound given there for the q -Gevrey order of the solution coincides with the one described here in cases (RS-R) and (IS), provided $h_n = 1$. In the other cases, Zhang proves that some bound exists but without a detailed control. In particular, our bound in case (RS-N) is more accurate because we prove that the solution is of q -Gevrey order s , for any $s > 0$. If $h_n = 1$,

the bound found in [33] is described with the aid of the Newton-Adams Polygon (see [1, 2]) of the linearized operator along $y(x)$:

$$L_y = \sum_{j=0}^n \frac{\partial P}{\partial Y_j} [y(x)] \sigma^j \in \mathbb{C}[[x]][\sigma].$$

By Proposition 1, we know that L_y is not identically zero if and only if the pivot point of P with respect to $y(x)$ has ordinate one. The Newton-Adams Polygon $\mathcal{N}_q(L_y)$ of L_y is defined as follows: for each $0 \leq j \leq n$, let $l_j = \text{ord}_{\frac{\partial P}{\partial Y_j}}[y(x)] \in \mathbb{N} \cup \{\infty\}$. Notice that $l_j = a_j$ if $h_j = 1$. Then $\mathcal{N}_q(L_y)$ is the convex hull of the set $\{(j, l_j + r) \mid l_j \neq \infty, r \geq 0\}$. It is easy to check that $s(P; y(x))$ is the reciprocal of the minimum of the positives slopes of $\mathcal{N}_q(L_y)$.

Remark 7. The labels (RS-*) and (IS-*) in Definition 6 correspond to the singularity type of the linearized operator L_y (regular or irregular). The labels (*-R) or (*-N) denote whether the solution $y(x)$ is a regular solution of P (i.e. $h_n = 1$) or not.

4.1. Reduction to solved form. In order to prove Theorem 4, we first show (in the paragraphs below) that we may assume that the equation $P[y] = 0$ is in solved form and that the general and all the relative pivot points with respect to the variables Y_j are reached at step 0.

Let $y(x) = \sum_{i=0}^{\infty} c_i x^i \in \mathbb{C}[[x]]$ be a solution of $P[y] = 0$. We apply the process described in the proof of Lemma 7 to P and $y(x)$ in three steps:

- (a) Replace P by some of its derivatives H such that the ordinate of the pivot point of H with respect to $y(x)$ is equal to 1 and $s(P; y(x)) = s(H; y(x))$.
- (b) Let N be large enough so that all the relative points $Q_j(H; y(x))$, for $0 \leq j \leq n$, have been reached at step $N - 1$.
- (c) Let $\gamma = N - 1$ and consider $P^* = \gamma H_N$ and $y^*(x) = \sum_{i=N}^{\infty} c_i x^{i-N+1}$. Then, $P^*[y] = 0$ is in quasi-solved form and $P^*[y^*(x)] = 0$.

If $\bar{y}(x) = \sum_{i=N}^{\infty} c_i x^i$, then the relative pivot points of $y(x)$ with respect to H are the same as the relative pivot points of $\bar{y}(x)$ with respect to H_N . Hence, $s(H; y(x)) = s(H_N; \bar{y}(x))$. Finally, by the properties described in subsection 3.1.1, $\tau(Q_j(H_N; \bar{y}(x))) = Q_j(P^*; y^*(x))$, where τ is a plane affine map whose restriction to the line of points with ordinate one is a translation, so that $s(H_N; \bar{y}(x)) = s(P^*; y^*(x))$. This proves that $s(P; y(x)) = s(P^*; y^*(x))$. Moreover, the general and relative pivot points $Q_j(P^*; y^*(x))$ are reached at step 0. It is straightforward to prove that if P is of q -Gevrey order t , then H , H_N and P^* are all of q -Gevrey order t . Also $y^*(x)$ and $y(x)$ have the same q -Gevrey order. This shows that it is enough to prove Theorem 4 when the q -difference equation $P[y] = 0$ is in quasi-solved form and the relative pivot points $Q_j(P; y(x))$ are reached at step 0.

Finally, assuming that $P[y] = 0$ is in quasi-solved form, since $|q| > 1$, the set Σ^+ is finite. Let N be an integer greater than the maximum of Σ^+ , $P^* = {}^N(P_{N+1})$ and $y^*(x) = \sum_{i=N+1}^{\infty} c_i x^{i-N}$. It is clear that $s(P; y(x)) = s(P^*; y^*(x))$, and also that P^* and $y^*(x)$ are of the same q -Gevrey order as P and $y(x)$ respectively. From this we conclude that we may assume the q -difference equation $P[y] = 0$ is in solved form.

4.2. Recursive formula for the coefficients. Let $y(x) = \sum_{i=1}^{\infty} c_i x^i$ be a power series solution of the q -difference equation $P[y] = 0$, where

$$P = \sum_{(\alpha, \rho) \in \mathbb{N} \times \mathbb{N}^{n+1}} P_{\alpha, \rho} x^\alpha Y^\rho \in \mathbb{C}[[x, Y_0, Y_1, \dots, Y_n]].$$

Assume that it is in solved form and that the general pivot point Q with respect to $y(x)$ and the relative ones $Q_j = (a_j, h_j)$ are all reached at step 0. Since the equation is in solved form, $Q = (0, 1)$. Let r be the maximum index j , $0 \leq j \leq n$, such that $Q_j = Q$ and let $\Psi(T)$ be the indicial polynomial of P at point Q . From equation (11) one has

$$(13) \quad \Psi(q^i) c_i = -\text{Coeff}(P_i; x^i Y^0), \quad i \geq 1.$$

As usual $P_i = P[c_1 x + \dots + c_{i-1} x^{i-1} + Y]$. We are interested in computing $\text{Coeff}(P_i; x^i Y^0)$ in terms of c_1, c_2, \dots, c_{i-1} . To this end, we shall consider formal series H^i in the variables $T_{\alpha, \rho}$, $C_{j, l}$, x and Y_0, Y_1, \dots, Y_n , where $\alpha \in \mathbb{N}$, $\rho = (\rho_0, \dots, \rho_n) \in \mathbb{N}^{n+1}$, $0 \leq j \leq n$ and $1 \leq l \leq i-1$, defined as follows

$$H^i = \sum_{(\alpha, \rho) \in \mathbb{N}^{n+2}} T_{\alpha, \rho} x^\alpha \prod_{0 \leq j \leq n} \left(\sum_{l=1}^{i-1} C_{j, l} x^l + Y_j \right)^{\rho_j}.$$

For $(\beta, \gamma) \in \mathbb{N} \times \mathbb{N}^{n+1}$, let $H_{\beta, \gamma}^i$ be the coefficient of $x^\beta Y^\gamma$ in H^i . It is a polynomial with coefficients in \mathbb{N} and in the variables $T_{\alpha, \rho}$ and $C_{j, l}$. Denote $L_i = H_{i, \underline{0}}^i$, i.e. the coefficient of $x^i Y^0$ in H^i . A simple computation shows that

$$L_i = \sum_{(\alpha, \rho, \underline{d}) \in \mathcal{F}_i} B_{\alpha, \rho, \underline{d}}^i T_{\alpha, \rho} \prod_{0 \leq j \leq n} \prod_{1 \leq l \leq i-1} C_{j, l}^{d_{j, l}},$$

where $B_{\alpha, \rho, \underline{d}}^i$ are non-negative integers and the summation set \mathcal{F}_i comprises those $(\alpha, \rho, \underline{d})$, such that $\alpha \in \mathbb{N}$, $\rho \in \mathbb{N}^{n+1}$, $\underline{d} = (d_{j, l}) \in \mathbb{N}^{(n+1)(i-1)}$, for $0 \leq j \leq n$, $1 \leq l \leq i-1$, for which the following formulæ hold:

$$(14) \quad \alpha + \sum_{j, l} l d_{j, l} = i,$$

$$(15) \quad \sum_l d_{j, l} = \rho_j, \quad \text{and so,} \quad \sum_{j, l} d_{j, l} = |\rho|.$$

Remark 8. Notice that, substituting in H^i the variables $T_{\alpha, \rho}$ for $P_{\alpha, \rho}$ and $C_{j, l}$ for $c_{j, l} := q^{j^l} c_l$, one obtains P_i . Hence,

$$\text{Coeff}(P_i; x^i Y^0) = L_i(P_{\alpha, \rho}, c_{j, l}).$$

However, in order to have an optimal control on the q -Gevrey growth, we need to be more precise and use the position of the relative pivot points of P with respect to $y(x)$, and refine the summation set: let \mathcal{F}'_i be the subset of \mathcal{F}_i composed by those $(\alpha, \rho, \underline{d})$ satisfying the following properties:

$$(16) \quad \text{If } j > r, h_j \geq 2, \text{ and } l > i/2 \text{ then } d_{j,l} = 0.$$

$$(17) \quad \text{If } j > r, h_j = 1, \text{ and } l > i - a_j \text{ then } d_{j,l} = 0.$$

Let \mathcal{F}''_i be the complement of \mathcal{F}'_i in \mathcal{F}_i and let L'_i (resp. L''_i) be the sum of those terms in L_i corresponding to those $(\alpha, \rho, \underline{d})$ in \mathcal{F}'_i (resp. in \mathcal{F}''_i); obviously $L_i = L'_i + L''_i$.

Lemma 9. *The following equality holds: $\text{Coeff}(P_i; x^i Y^0) = L'_i(P_{\alpha, \rho}, c_{j,l})$.*

Proof. Take l_0 with $1 \leq l_0 \leq i - 1$, and consider $P_{l_0} = P[\sum_{l=1}^{l_0-1} c_l x^l + Y]$ (see (3)). Let \bar{P}_{l_0} be the series obtained substituting in P_{l_0} the expression $\sum_{l=l_0}^{i-1} C_{j,l} x^l + Y_j$ for the variable Y_j , $0 \leq j \leq n$. By construction, $\bar{P}_{l_0} = P_i$, hence $L_i = \text{Coeff}(\bar{P}_{l_0}; x^i Y^0)$.

Write $P_{l_0} = \sum_{(\alpha, \rho) \in \mathbb{N} \times \mathbb{N}^{n+1}} (P_{l_0})_{\alpha, \rho} x^\alpha Y^\rho$. Expanding \bar{P}_{l_0} as a series in the variables $C_{j,l}$, $l_0 \leq l \leq i - 1$, x and Y_j , $0 \leq j \leq n$, let us denote, for $r < j_0 \leq n$, by A_{j_0, l_0} the sum of terms of \bar{P}_{l_0} in which the variable C_{j_0, l_0} appears effectively. In order to compute A_{j_0, l_0} , it is only necessary to take into account the terms of P_{l_0} in which the variable Y_{j_0} appears effectively, that is, only consider the sum over the indices $(\alpha, \rho) \in \mathcal{C}_{j_0}(P_{l_0})$. Since we are assuming that the pivot point $Q_{j_0} = (a_{j_0}, h_{j_0})$ of P with respect to $y(x)$ relative to the variable Y_{j_0} is reached at step 0, we may assume that the order in x of A_{j_0, l_0} is greater than or equal to $a_{j_0} + h_{j_0} l_0$. If j_0, l_0 satisfy the premise of either (16) or (17), then $a_{j_0} + h_{j_0} l_0 > i$ and the variable C_{j_0, l_0} does not appear effectively in the coefficient of $x^i Y^0$ in \bar{P}_{l_0} . From this one infers that $L''_i(P_{\alpha, \rho}, c_{j,l}) = 0$. \square

From the definition of r , one has $\Psi(T) = P_{0, e_0} + P_{0, e_1} T + \dots + P_{0, e_r} T^r$, with $P_{0, e_r} \neq 0$. In particular, $\Psi(T)$ has degree r . Moreover, since the equation $P[y] = 0$ is in solved form, $\Psi(q^i) \neq 0$, for $i \geq 1$. From equation (13) and Lemma 9, the following recursive formula holds for all $i \geq 1$:

$$(18) \quad c_i = \frac{-1}{\Psi(q^i)} L'_i(P_{\alpha, \rho}; c_{j,l}),$$

where $c_{j,l} = q^{j \cdot l} c_l$, $1 \leq l \leq i - 1$ and $0 \leq j \leq n$.

4.3. A majorant series. Assume the hypotheses and notations of the previous sub-section and that P has q -Gevrey order $t \geq 0$. Let $s = s(P; y(x))$. Consider the equation in two variables x and w :

$$(19) \quad w = |q|^{-\frac{s+t}{2}} |c_1| x + \sum_{(\alpha, \rho) \in \mathcal{C}'} G_{\alpha, \rho} x^\alpha w^{|\rho|},$$

where $G_{\alpha, \rho} = |P_{\alpha, \rho}| |q|^{-\frac{t}{2}(\alpha + |\rho|)^2 + k_1(\alpha + |\rho|) + k_2}$, k_1 and k_2 are positive constants to be specified later, and \mathcal{C}' is the set $\mathbb{N} \times \mathbb{N}^{n+1}$ without the points

$(0, \underline{0})$, $(1, \underline{0})$ and $(0, e_j)$ for $0 \leq j \leq n$. It is straightforward to prove that the right hand side of (19) is a convergent series and that the equation has a unique power series solution $w(x) = \sum_{i=1}^{\infty} c'_i x^i$, whose coefficients c'_i satisfy the recursive formulae:

$$c'_1 = |q|^{-\frac{s+t}{2}} |c_1|, \quad c'_i = L_i(G_{\alpha, \rho}; \{c'_{j,l}\}), \quad i \geq 2,$$

where $c'_{j,l} = c'_l$, for $1 \leq l \leq i-1$, and $0 \leq j \leq n$. In particular, $c'_i \geq 0$, for all $i \geq 1$, since the coefficients of L_i are non-negative. By Puiseux's theorem, the series $w(x)$ is convergent. The following lemma finishes the proof of Theorem 4.

Lemma 10. *With the above notations, there exist positive constants k_1 and k_2 such that the coefficients c'_l of the solution of equation (19) satisfy*

$$(20) \quad |c_l| \leq |q|^{\frac{s+t}{2}l^2} |c'_l|, \quad l \geq 1.$$

Proof. The above inequality holds trivially for $l = 1$. Assume that it holds for $l = 1, 2, \dots, i-1$. Using equation (18) and the fact that the coefficients of L_i are non-negative, one gets

$$\begin{aligned} |c_i| &\leq \frac{1}{|\Psi(q^i)|} \sum_{(\alpha, \rho, \underline{d}) \in \mathcal{F}'_i} B_{\alpha, \rho, \underline{d}}^i |P_{\alpha, \rho}| \prod_{j,l} (|q|^{jl} |c_l|)^{d_{j,l}} \\ (21) \quad &\leq \frac{1}{|\Psi(q^i)|} \sum_{(\alpha, \rho, \underline{d}) \in \mathcal{F}'_i} B_{\alpha, \rho, \underline{d}}^i G_{\alpha, \rho} \frac{|q|^{\frac{t}{2}(\alpha+|\rho|)^2}}{|q|^{k_1(\alpha+|\rho|)+k_2}} \prod_{j,l} \left(|q|^{jl} |q|^{\frac{s+t}{2}l^2} |c'_l| \right)^{d_{j,l}} \\ &= \sum_{(\alpha, \rho, \underline{d}) \in \mathcal{F}'_i} R_i(\alpha, \rho, \underline{d}) B_{\alpha, \rho, \underline{d}}^i G_{\alpha, \rho} \prod_{j,l} |c'_l|^{d_{j,l}}, \end{aligned}$$

where the indices j and l are $0 \leq j \leq n$ and $1 \leq l \leq i-1$, and

$$\begin{aligned} R_i(\alpha, \rho, \underline{d}) &= \frac{1}{|\Psi(q^i)|} |q|^{r_i(\alpha, \rho, \underline{d})}, \\ r_i(\alpha, \rho, \underline{d}) &= \sum_{j,l} \left(jl + \frac{s+t}{2} l^2 \right) d_{j,l} + \frac{t}{2} (\alpha + |\rho|)^2 - k_1(\alpha + |\rho|) - k_2. \end{aligned}$$

Claim (proved below): there exist positive constants k_1 and k_2 , such that

$$(22) \quad R_i(\alpha, \rho, \underline{d}) \leq |q|^{\frac{s+t}{2}i^2}, \quad (\alpha, \rho, \underline{d}) \in \mathcal{F}'_i.$$

Assuming the claim and using equations (21) and (22), one gets

$$|c_i| \leq |q|^{\frac{s+t}{2}i^2} \sum_{(\alpha, \rho, \underline{d}) \in \mathcal{F}'_i} B_{\alpha, \rho, \underline{d}}^i G_{\alpha, \rho} \prod_{j,l} |c'_l|^{d_{j,l}} = |q|^{\frac{s+t}{2}i^2} L'_i(G_{\alpha, \rho}; \{|c'_l|\}).$$

Since the coefficients of L_i , the elements $G_{\alpha, \rho}$ and c'_l are all non-negative real numbers, then $L'_i(G_{\alpha, \rho}; \{|c'_l|\}) \geq 0$. Hence,

$$L'_i(G_{\alpha, \rho}; \{|c'_l|\}) \leq L'_i(G_{\alpha, \rho}; \{c'_l\}) + L''_i(G_{\alpha, \rho}; \{c'_l\}) = L_i(G_{\alpha, \rho}; \{c'_l\}) = |c'_i|,$$

which proves the Lemma. \square

Proof of Claim. Since the degree of $\Psi(T)$ is r , $|q| > 1$ and $\Psi(q^i) \neq 0$ for $i \geq 1$, there exists a constant $K_1 > 0$, such that $|q|^{ir} \leq K_1 |\Psi(q^i)|$, for all $i \geq 1$. Thus, it is enough to prove that there exist $k_1 > 0$ and $k_2 > 0$ such that $r_i(\alpha, \rho, \underline{d}) \leq \frac{s+t}{2}i^2 + ri$, for all $i \geq 1$ and all $(\alpha, \rho, \underline{d}) \in \mathcal{F}'_i$. Grouping the terms of r_i and rearranging, we divide the inequality above into two parts so that it is enough to prove the existence of positive constants k_1 and k_2 , such that for all $(\alpha, \rho, \underline{d}) \in \mathcal{F}'_i$ and $i \geq 1$, the following inequalities hold:

$$(23) \quad \frac{s}{2} \sum_{j,l} l^2 d_{j,l} + \sum_{j,l} j l d_{j,l} \leq \frac{s}{2} i^2 + r i + k_2,$$

$$(24) \quad \frac{t}{2} \sum_{j,l} l^2 d_{j,l} + \frac{t}{2} (\alpha + |\rho|)^2 \leq \frac{t}{2} i^2 + k_1 (\alpha + |\rho|) + k_2.$$

We first prove the existence for (23) and then for (24).

Proof of inequality (23). Call $r'_i(\alpha, \rho, \underline{d})$ the left hand side of (23). Let $\mathcal{F}'_i = F_1 \cup F_2$, where F_1 is the subset formed by those $(\alpha, \rho, \underline{d})$ such that $l > i/2$ implies $d_{j,l} = 0$, and F_2 is its complement in \mathcal{F}'_i . We shall bound r'_i in each of F_1, F_2 by a polynomial $\bar{r}'(i) = \bar{r}'_2 i^2 + \bar{r}'_1 i + \bar{r}'_0$, such that, either $\bar{r}'_2 < \frac{s}{2}$ or $\bar{r}'_2 = \frac{s}{2}$ and $\bar{r}'_1 \leq r$. Adjusting k_2 conveniently, one gets (23).

Let $(\alpha, \rho, \underline{d}) \in F_1$. This implies that if $d_{j,l} \neq 0$, then $l \leq i/2$. As $j \leq n$, and $\sum_{j,l} l d_{j,l} \leq i$ (which follows from (14)), we conclude that

$$r'_i = \frac{s}{2} \sum_{j,l} l^2 d_{j,l} + \sum_{j,l} j l d_{j,l} \leq \frac{s}{4} \sum_{j,l} l d_{j,l} + n \sum_{j,l} l d_{j,l} \leq \frac{s}{4} i^2 + n i = \bar{r}'(i).$$

If $s \neq 0$, then $\bar{r}'_2 < s/2$. Otherwise, $s = 0$, and by Definition 6, $r = n$, hence $\bar{r}'_1 \leq r$. This proves that the polynomial $\bar{r}(i)$ satisfies our requirements.

Let $(\alpha, \rho, \underline{d}) \in F_2$. There exists a pair (j_0, l_0) such that $l_0 > i/2$ and $d_{j_0, l_0} \neq 0$. By inequality (14), this pair is unique and $d_{j_0, l_0} = 1$. In this case, equation (14) reads as

$$(25) \quad \alpha + \sum_{j,l \neq l_0} l d_{j,l} + l_0 = i, \quad \text{and in particular} \quad \sum_{j,l \neq l_0} l d_{j,l} \leq a,$$

where $a = i - l_0 < i/2$. This implies also that for $l \neq l_0$ and $d_{j,l} \neq 0$ one has $l \leq a$. Therefore,

$$\begin{aligned} r'_i &= \frac{s}{2} \left(\sum_{j,l \neq l_0} l^2 d_{j,l} + l_0^2 \right) + \sum_{j,l \neq l_0} j l d_{j,l} + j_0 l_0 \\ &\leq \frac{s}{2} \left(a \sum_{j,l \neq l_0} l d_{j,l} + (i - a)^2 \right) + n \sum_{j,l \neq l_0} l d_{j,l} + j_0(i - a) \\ &\leq \frac{s}{2} (2a^2 - 2a i + i^2) + n a + j_0(i - a) \\ &= (s/2)i^2 + (j_0 - s a)i + (s a^2 + n a - a j_0) := f_i(a). \end{aligned}$$

For a fixed i , the graph of $f_i(a)$ is an upwards parabola, so its maximum in an interval is reached at its endpoints. The available range for a depends on j_0 . If $j_0 \leq r$, then $a \in [1, i/2]$, and we take $\bar{r}'(i) = s/2 i^2 + r i + \bar{r}_0$. We can chose \bar{r}'_0 in such a way that $\max\{f_i(1), f_i(i/2)\} \leq \bar{r}'(i)$, for all $i \geq 1$, because the coefficient of i^2 in $f_i(i/2)$ is $s/4 < s/2$ and the coefficient of i in $f_i(1)$ is $j_0 - s \leq r$. If, on the other hand, $j_0 > r$, case (16) does not happen because $l_0 > i/2$ so that $h_{j_0} = 1$ and $l_0 \leq i - a_{j_0}$, and the range for a is $[a_{j_0}, i/2]$. By definition of s , one has $j_0 - s a_{j_0} \leq r$ and $s > 0$. Consider $\bar{r}'(i) = s/2 i^2 + r i + \bar{r}'_0$, where \bar{r}'_0 is chosen in such a way that $\max\{f_i(i/2), f_i(a_j); j > r, h_j = 1\} \leq \bar{r}(i)$, for all $i \geq 1$. Such an \bar{r}'_0 exists because the coefficient in $f_i(i/2)$ of i^2 is $s/4 < s/2$, $f_i(a_j) \leq (s/2)i^2 + (j - s a_j)i + s a_j^2 + n a_j$ and $j - s a_j \leq r$ for those j such that $j > r$ and $h_j = 1$.

Proof of inequality (24). For $t = 0$, the inequality holds trivially, so we may assume that $t > 0$. Let $(\alpha, \rho, \underline{d}) \in \mathcal{F}'_i$. Denote $d_l = \sum_{j=0}^n d_{j,l}$, for $1 \leq l \leq i-1$ and let l_0 be the maximum of the indices l such that $d_l \neq 0$. From equations (14) and (15), the fact that $l_0 \geq 1$ and $d_{l_0} \geq 1$, one gets:

$$i - |\rho| = \alpha + \sum_l l d_l - \sum_l d_l = \alpha + \sum_{l \neq l_0} (l-1) d_l + (l_0-1) d_{l_0} \geq \alpha + l_0 - 1.$$

From which $i - l_0 \geq \alpha + |\rho| - 1$. Taking into account that $\alpha \geq 0$, equation (14), and the fact that $l_0 \geq l$ for any l with $d_l \neq 0$, we conclude that

$$\begin{aligned} i^2 &= (i - l_0 + l_0)^2 = (i - l_0)^2 + l_0^2 + 2 l_0 (i - l_0) \\ &\geq (\alpha + |\rho| - 1)^2 + l_0^2 + 2 l_0 \left(\sum_{l \neq l_0} l d_l + l_0 (d_{l_0} - 1) \right) \\ &\geq (\alpha + |\rho| - 1)^2 + l_0^2 + \sum_{l \neq l_0} l^2 d_l + l_0^2 (d_{l_0} - 1) \\ &\geq (\alpha + |\rho|)^2 - 2(\alpha + |\rho|) + \sum_l l^2 d_l. \end{aligned}$$

This gives inequality (24) for $k_1 \geq 2/t$ and finishes the proof of Theorem 4. \square

5. WORKING EXAMPLE

Let us consider the q -difference equation $P[y] = 0$ of order 5 and degree 6, where

$$P = 4Y_1^4 - 9Y_0^2 Y_1 Y_2 + 2Y_0^3 Y_2 - x^3 Y_0^4 Y_5^2 + \frac{x Y_0 Y_2}{q^4} - \frac{x^3 Y_2}{q^4} - x^3 Y_0 + x^5,$$

and $q = 4$. Its Newton Polygon is $\mathcal{N}(P)$ in Figure 1. It has four vertices $v_0 = (3, 6), v_1 = (0, 4), v_2 = (1, 2), v_3 = (5, 0)$ and three sides L_1, L_2 and L_3 with respective co-slopes $\gamma_1 = -3/2, \gamma_2 = 1/2$ and $\gamma_3 = 2$. We apply some steps of Procedure 1 to P . As P is a polynomial, $\mu_{-1}(P) = -\infty$.

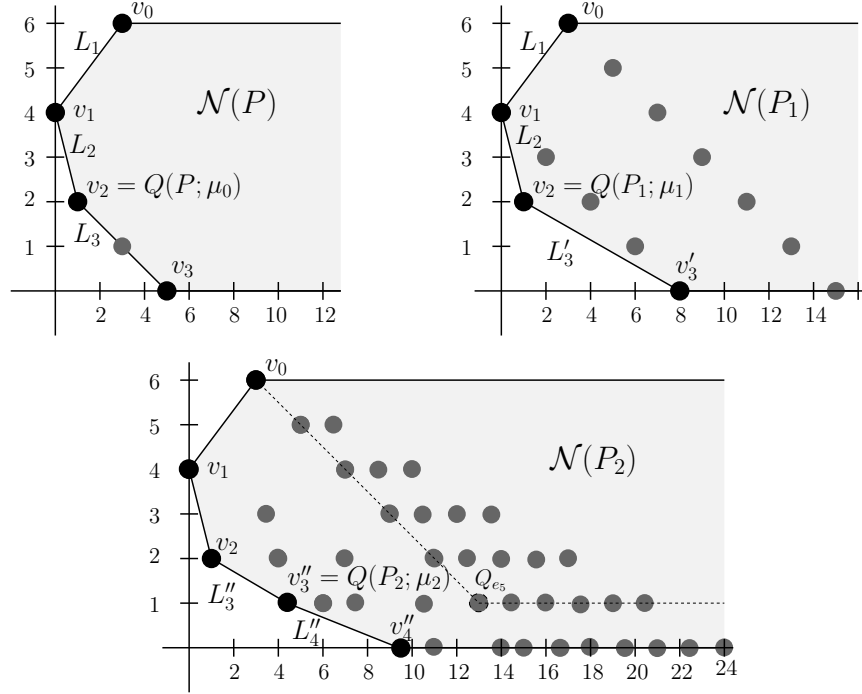


FIGURE 1. Newton Polygons from the working example.

In order to find all the possible starting terms $c_0 x^{\mu_0}$ of a solution, we need to consider all the vertices and sides of $\mathcal{N}(P)$ according as formulæ (5) and (6). For the vertices, we get: $\Psi_{(P;v_0)}(T) = -3T^{10}$, $\Psi_{(P;v_1)}(T) = T^2(T-2)(4T-1)$, $\Psi_{(P;v_2)}(T) = T^2/q^4$, $\Psi_{(P;v_3)}(T) = 1$. Hence, for $j = 0, 1, 2, 3$ the only satisfiable formula in (6) is the one corresponding to vertex v_1 , that is $\Psi_{(P;v_1)}(q^\mu) = 0$ and $-3/2 < \mu < 1/2$. This gives $\mu = -1$ for any nonzero c . For the sides, we get: $\Phi_{(P;\gamma_1)}(c) = q^{-15}c^4(2q^{12} - 9q^{21/2} + 4q^9 - c^2)$, $\Phi_{(P;\gamma_2)}(c) = c^2/64$, and $\Phi_{(P;\gamma_3)}(c) = (c-1)^2$. According as (5), the only possible starting terms related to the sides are $\pm 1024\sqrt{15}x^{-3/2}$ and x^2 . Notice that L_2 gives rise to no starting term.

Following Procedure 1 we choose x^2 , that is $c_0 = 1$ and $\mu_0 = 2$. The polynomial $P_1 = P[x^2 + Y]$ has 33 terms that we do not exhibit; its Newton Polygon is $\mathcal{N}(P_1)$ in Figure 1. Since $y = 0$ is not a solution of $P_1[y] = 0$ because $\mathcal{C}(P_1)$ has points on the OX -axis, we need to perform step (a.2) of Procedure 1, that is finding $\mu > \mu_0 = 2$ and $c \neq 0$ so that $\Psi_{(P_1;\mu)}(c) = 0$. Thus, we can only use the vertices v_2 and v'_3 and side L'_3 .

For formula (6) we get $\Psi_{(P_1;v_2)}(T) = \Psi_{(P;v_2)}$ and that $\Psi_{(P_1;v'_3)}(T)$ is a constant, hence those vertices do not give rise to subsequent terms. For side L'_3 , we get $\mu_1 = 7/2$ and $\Psi_{(P_1;\mu_1)}(c) = 64c^2 + 225792$, so that there are two possibilities for c_1 . We choose $c_1 = 21\sqrt{8}\sqrt{-1}$ and go on with Procedure 1.

Let us consider $P_2 = P_1[c_1x^{\mu_1} + Y]$ whose Newton Polygon is $\mathcal{N}(P_2)$ having a side L_3'' of the same co-slope as L_3' and another L_4'' of co-slope 5. As vertex v_3'' gives $\Psi_{(P_2;v_3'')}(q^\mu) = q^{2\mu} + 16384$ which has no real solutions, it is useless to find μ_2 . Hence we must use L_4'' which gives $\mu_2 = 5$ and (after a trivial computation) $c_2 = -88984/65$.

Notice that, after performing the first two steps detailed above and getting $x^2 + 21\sqrt{8}\sqrt{-1}x^{7/2}$, the fact that v_3'' gives rise to a formula which no $\mu > 7/2$ can satisfy and that it has ordinate one implies that, taking $P^* = {}^{\mu_1}P_2$, the equation $P^*[y] = 0$ is solved form. Therefore, by Lemma 8 there exists a unique solution of $P[y] = 0$ of the form:

$$y(x) = x^2 + 21\sqrt{8}\sqrt{-1}x^{7/2} + o(x^{7/2}).$$

Notice also that as $P^* \in \mathbb{C}[[x^{1/2}]] [Y]$, Lemma 8 guarantees as well that $y(x) \in \mathbb{C}[[x^{1/2}]]$.

The pivot point of P with respect to $y(x)$ is $Q(y(x); P) = v_3'' = (4.5, 1)$ and Y_2 is the highest order appearing effectively in it, hence $r = 2$ in Definition 5. There being no monomials with Y_3 or Y_4 in P we only need consider the pivot point relative to Y_5 which is the point $Q_{e_5}(y(x); P) = (13, 1)$ (notice that $\mathcal{C}_{e_5}(P_2)$ is in the region above and to the right of the dashed line). Applying Definition 5 formally we would get $s(y(x); P) = \frac{5-2}{13-4.5} = 6/17$.

Concerning the growth of the coefficients of $y(x)$, we transform it into a formal power series in order to apply Theorem 4. We do this by means of the ramification $x = t^2$. The series $y(t)$ is a solution of a \bar{q} -difference equation $\bar{P}[y] = 0$ derived from P with $\bar{q} = q^{1/2}$. The ramification induces a horizontal homothecy of ratio 2 on the cloud of points of P , P_1 and P_2 . Hence $s(y(t); \bar{P}) = \frac{5-2}{2(13-4.5)} = 3/17$ is a bound for the \bar{q} -Gevrey order of $y(t)$.

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